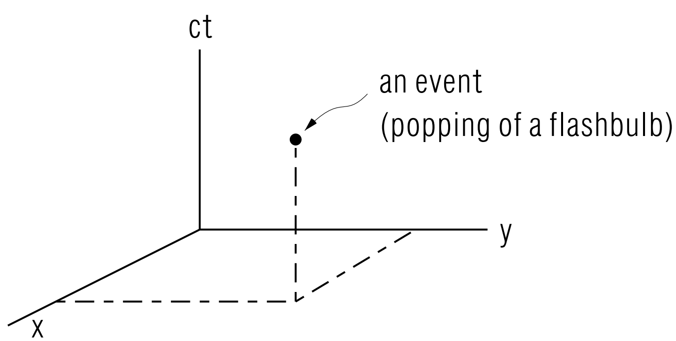
MISN-0-75

RELATIVISTIC SPACE-TIME: FOUR-VECTORS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

RELATIVISTIC SPACE-TIME: FOUR-VECTORS

by

Frank Zerilli

Title: **Relativistic Space-Time: Four-Vectors**

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Version: 2/1/2000 Evaluation: Stage 0

Length: 1 hr; 20 pages

Input Skills:

1. Write a given system of linear equations in matrix form (MISN-0-301).
2. State down the Lorentz transformation and use it to transform given positions and times (MISN-0-12).
3. Define relativistic energy and momentum in terms of mass and velocity (MISN-0-24).

Output Skills (Knowledge):

K1. Vocabulary: event, four-vector, world line.

K2. Write the Lorentz transformation in matrix form.

K3. Show that relativistic energy and momentum are the components of a four-vector.

Output Skills (Problem Solving):

S1. Transform the coordinates of a given event from one inertial frame to another.

S2. Transform given energy-momentum four-vectors from one given Lorentz (inertial) frame to another.

Post-Options:

1. "Topics in Relativity: Doppler Effect and Pair Production" (MISN-0-308).

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1. Description

1a. Events in Spacetime. The term “event” is used to specify a point in what we will call spacetime which consists of the usual three-dimensional euclidean space of location plus the one-dimensional axis of time. As an example, suppose we fire a flashbulb (see Fig. 1). The firing of the flashbulb is an event and to specify its location in spacetime we must give four quantities - three quantities (for example, rectangular coordinates x, y, z in some inertial frame) which specify its location in space, plus one quantity which specifies its “location” in time (in some inertial frame).

1b. Four-vectors. The four quantities constitute a vector called a “four-vector” in this four-dimensional space. We will, for convenience, choose to use (ct) as the fourth component of the four-vector rather than t, since the equations we will write are then a little simpler. Also, the four components of the vector then all have the same dimension: length. Note: (ct) is the distance light travels in time t.

1c. World Lines. The trajectory of a point object in spacetime is called its “world line.” For example, the world line of the point object stationary in the frame described by coordinates (x, y, z, ct) is a straight line parallel to the t-axis and could be described by the set of equations:

$$x(t) = x_0$$

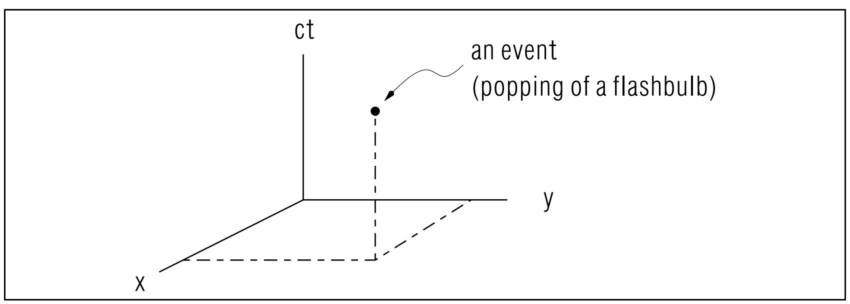

Figure 1. An “event” in spacetime.

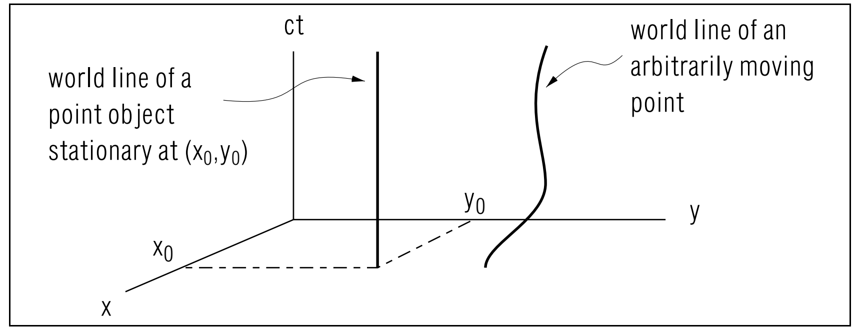

Figure 2. “World lines” in spacetime.

$$y(t) = y_0$$

$$z(t) = z_0$$

$$ct = ct$$

where x_0, y_0, z_0 are the space coordinates of the stationary point object. Or, we can say that the four-vector of the point object at any time t is:

$$(x_0, y_0, z_0, ct)$$

In general, a world line can be specified by giving a four-vector which is a function of time, or some other parameter, call it τ:

$$[\, x(\tau), y(\tau), z(\tau), ct(\tau)\,].$$

All physical objects have some extension: we speak of the length, width, and height of objects. Upon reflection we see that for a complete specification of the extension of an object we should specify not only its spatial dimensions but also its extension in time. For example, we might specify the dimension of the building as: 100 ft long, 50 ft wide, 30 ft high, and 20 years extended in time (it was built in 1910 and demolished in 1930). In a spacetime diagram, we might describe the object by giving the world lines for each point of the object (see Fig. 3).

2. The Lorentz Transformation as a Matrix

We know that the space and time coordinates of a point with respect to one Lorentz (inertial) frame are related to the coordinates with respect to a second Lorentz (inertial) frame by Lorentz transformation. Let L be a Lorentz frame with coordinates x, y, z, ct and let L′ be a Lorentz

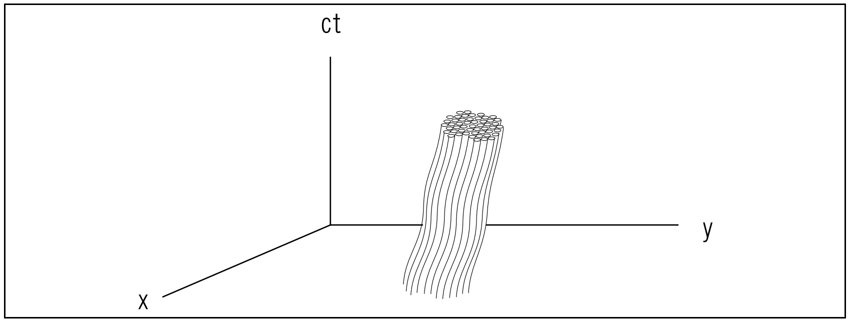

Figure 3. A collection of world lines for an extended object.

frame with coordinate axes x', y', z', ct' moving uniformly with velocity v parallel to the z-axis of L. Also let the x', y', z' axes of L′ be parallel to the x, y, z axes, respectively, of L. Then the Lorentz transformation may be written:

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma z - \beta\gamma ct \\ ct' &= -\beta\gamma z + \gamma ct \end{aligned}$$

where we have utilized the widely used notation:

$$\beta \equiv \frac{v}{c}; \qquad \gamma \equiv (1-\beta^2)^{-1/2}.$$

This may be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

Since the x and y coordinates are not affected in this case we may suppress them and write:

$$\begin{pmatrix} z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} z \\ ct \end{pmatrix}$$

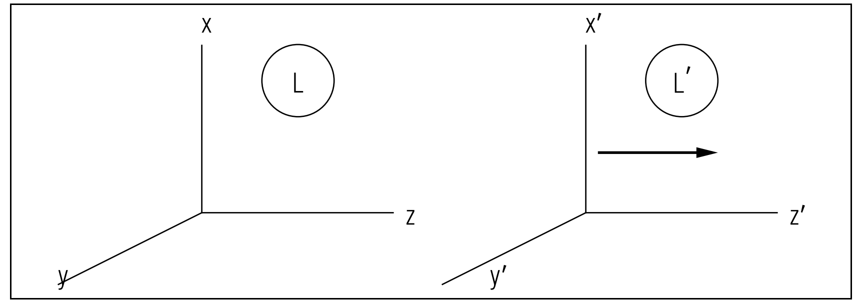

Figure 4. Two inertial frames and a relative velocity.

3. Precise Definition of a 4-Vector

Any set of four quantities which transforms from one Lorentz frame to another just like the coordinates $(x,\, y,\, z,\, ct)$ of a point in spacetime is called a four-vector.[1] Thus $(x,\, y,\, z,\, ct)$ is a four-vector.

One example of a very important four-vector is the energy-momentum four-vector whose spatial components are the three components of relativistic momentum and whose fourth component is the relativistic energy (divided by c):

$$\left(p_x, p_y, p_z, \frac{E}{c}\right).$$

Knowing that the components of momentum and energy transform like a four-vector, we can find the momentum and energy of an object as viewed in any Lorentz frame if we are given the values in any one Lorentz frame. For example, suppose we have a particle of mass m at rest in one Lorentz frame. Consider a Lorentz frame moving with velocity v along the x-direction. Then, in this new frame, the particle is moving with velocity $-v$ along x'-axis and:

$$\begin{pmatrix} p'_x \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 \\ m_0 c \end{pmatrix} = \begin{pmatrix} -\beta\gamma m_0 c \\ \gamma m_0 c \end{pmatrix}$$

So:

$$p'_x = -\frac{m_0 v}{(1-v^2/c^2)^{1/2}} \qquad \text{and} \qquad E' = \frac{m_0 c^2}{(1-v^2/c^2)^{1/2}}.$$

[1] This definition is correct only for flat spacetime. In non-flat spacetimes, coordinates are not the components of four-vectors.

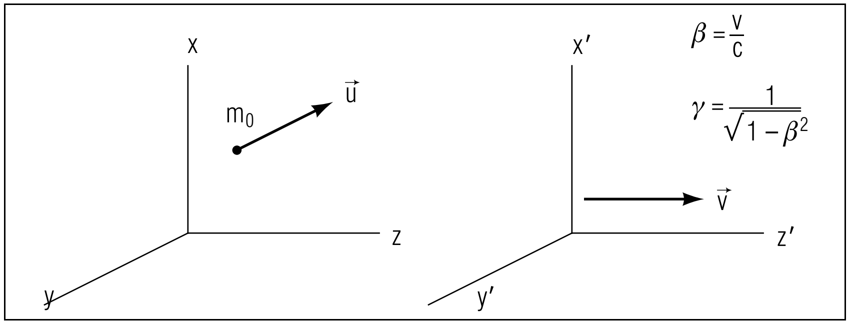

Figure 5. A velocity as seen from a stationary frame and from a moving frame.

This gives us just the expected values for momentum and energy of a particle of mass m_0 moving with velocity $-v$.

4. Proof of the Momentum-Energy 4-Vector

The most straightforward way of showing that momentum and energy are the components of a 4-vector is to write the momentum and energy of a particle in terms of its real mass and velocity in one Lorentz frame and then use the Lorentz transformation for velocity (that is, the velocity addition law) to find the form of the momentum and energy of the particle in a second Lorentz frame.

$$\vec{p} = \frac{m_0\vec{u}}{\sqrt{1-(u/c)^2}} \qquad \vec{p}\,' = \frac{m_0\vec{u}\,'}{\sqrt{1-(u'/c)^2}}$$
$$\frac{E}{c} = \frac{m_0 c}{\sqrt{1-(u/c)^2}} \qquad \frac{E'}{c} = \frac{m_0 c}{\sqrt{1-(u'/c)^2}}$$

Figure 5 shows a particle of mass m_0 moving with velocity $\vec{u}$ in the first frame. The second frame is moving at velocity v with respect to the first frame, along the x-axis, and the particle has a velocity $\vec{u}\,'$ as seen in the second frame.

As usual, let $\beta = v/c$ and $\gamma = \left(1-\beta^2\right)^{-1/2}$. The velocity addition law gives us:

$$u'_x = \frac{u_x - v}{1-\dfrac{u_x v}{c^2}} = \frac{u_x - \beta c}{1-\dfrac{u_x}{c}\beta}$$
$$u'_y = \frac{1}{\gamma}\,\frac{u_y}{1-\dfrac{u_x v}{c^2}} = \frac{1}{\gamma}\,\frac{u_y}{1-\dfrac{u_x}{c}\beta}$$

$$u'_z = \frac{1}{\gamma}\,\frac{u_z}{1-\dfrac{u_x v}{c^2}} = \frac{1}{\gamma}\,\frac{u_z}{1-\dfrac{u_x}{c}\beta}$$

The quantity $\sqrt{1-(u'/c)^2)}$ occurs several times in the energy and momentum formulas so let us calculate it in terms of the velocity in the first (unprimed) frame:

$$\begin{aligned}(u')^2 &= (u'_x)^2 + (u'_y)^2 + (u'_z)^2\\ &= \frac{(u_x-\beta c)^2 + \gamma^{-2}(u_y^2+u_z^2)}{\left(1-\dfrac{u_x}{c}\beta\right)^2}\end{aligned}$$

So:

$$\begin{aligned}1-\left(\frac{u'}{c}\right)^2 &= 1 - \frac{\left(\dfrac{u_x}{c}-\beta\right)^2 + \gamma^{-2}\left(\dfrac{u_y^2+u_z^2}{c^2}\right)}{\left(1-\dfrac{u_x}{c}\beta\right)^2}\\
&= \frac{1-\dfrac{u_x^2}{c^2}-\beta^2+\dfrac{u_x^2\beta^2}{c^2}-(1-\beta^2)\left(\dfrac{u_y^2+u_z^2}{c^2}\right)}{\left(1-\dfrac{u_x}{c}\beta\right)^2}\\
&= \frac{(1-\beta^2)\left(1-\dfrac{u_x^2}{c^2}\right)-(1-\beta^2)\left(\dfrac{u_y^2+u_z^2}{c^2}\right)}{\left(1-\dfrac{u_x}{c}\beta\right)^2}\\
&= \frac{1}{\gamma^2}\,\frac{1-\dfrac{u_x^2+u_y^2+u_z^2}{c^2}}{\left(1-\dfrac{u_x}{c}\beta\right)^2} = \frac{1}{\gamma^2}\,\frac{1-\dfrac{u^2}{c^2}}{\left(1-\dfrac{u_x}{c}\beta\right)^2}\end{aligned}$$

So, finally, we have the relation we want:

$$\frac{1}{\left[1-(u'/c)^2\right]^{1/2}} = \gamma\frac{1-\dfrac{u_x}{c}\beta}{\left[1-(u/c)^2\right]^{1/2}}$$

Now we can easily express the components of momentum and energy as seen in the second (primed) frame in terms of the values as seen in the first (unprimed) frame:

$$p_x = \frac{m_0 u_x}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \qquad \text{and} \qquad p'_x = \frac{m_0 u'_x}{\left(1 - \frac{u'^2}{c^2}\right)^{1/2}}$$

$$\begin{aligned} p'_x &= m_0 \gamma \frac{u_x - \beta c}{\left(1 - \frac{u_x}{c}\beta\right)} \frac{\left(1 - \frac{u_x}{c}\beta\right)}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \\ &= \frac{\gamma m_0 u_x - \gamma\beta m_0 c}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} = \gamma p_x - \beta\gamma \frac{E}{c} \end{aligned}$$

$$p_y = \frac{m_0 u_y}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \qquad \text{and} \qquad p'_y = \frac{m_0 u'_y}{\left(1 - \frac{u'^2}{c^2}\right)^{1/2}}$$

$$p'_y = m_0 \frac{1}{\gamma} \frac{u_y}{\left(1 - \frac{u_x}{c}\beta\right)} \frac{\gamma\left(1 - \frac{u_x}{c}\beta\right)}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} = p_y$$

$$p_z = \frac{m_0 u_z}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \qquad \text{and} \qquad p'_z = \frac{m_0 u'_z}{\left(1 - \frac{u'^2}{c^2}\right)^{1/2}}$$

$$p'_z = m_0 \frac{1}{\gamma} \frac{u_z}{\left(1 - \frac{u_x}{c}\beta\right)} \frac{\gamma\left(1 - \frac{u_x}{c}\beta\right)}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} = p_z$$

$$\frac{E}{c} = \frac{m_0 c}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \qquad \text{and} \qquad \frac{E'}{c} = \frac{m_0 c}{\left(1 - \frac{u'^2}{c^2}\right)^{1/2}}$$

$$\begin{aligned} \frac{E'}{c} &= \frac{m_0 c\gamma\left(1 - \frac{u_x}{c}\beta\right)}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \\ &= -\beta\gamma \frac{m_0 u_x}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} + \gamma \frac{m_0 c}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}} \\ &= -\beta\gamma p_x + \gamma \frac{E}{c}. \end{aligned}$$

If we write the above equations in matrix form we obtain:

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

This is precisely the equation for a Lorentz transformation along the x-axis. Thus, the components of momentum and energy transform exactly like the coordinates (x, y, z, ct) and so form a four-vector.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

Problems 10-11 also occur in this module's *Model Exam*.

1. A star 1000 light years away explodes at $t = 0$ (as described in our (earth) coordinate system). Write out the event vector (as column matrix). [Note that $c = 1$ is units of light years/year.]

2. A space ship is passing earth at $t = 0$. The space ship is moving (at $1/2\,c$) in the direction of the star (from problem 1). If the space ship observers also agree that $t = 0$ when the space ship passes earth, what is the event vector for the star explosion in the space ship's coordinate system?

3. An electron passes at $0.98\,c$ through an accelerator tube. The tube has length L_2 according to the electron. In the electron frame what are the event vectors for:

   a. the event where the electron enters the tube;

   b. the event where the electron leaves the tube. Note that x component is measured from the electron (i.e., with the vectors in the electron frame).

4. The above accelerator tube has length L_1 according to a stationary observer. Give the two event vectors corresponding to the two events of problem 3 in the stationary coordinate frame.

5. Referring again to problems 3 and 4, transform according to the Lorentz transformation events of problem 4 into the events of problem 3. Find a relation between L_2 and L_1. Note that the electron sees the accelerator tube to be contracted because it is moving relative to him.

6. An object follows a circular path in the $x - y$ plane. The radius of the circle is 6r and at $t = 0$, $x = r$, $y = 0$. It completes the circle in time T. Write the world line for this object.

7. a. Transform the world line of problem 6 to a frame moving at v in the $+z$ direction.

   b. In this frame the object moves in a helical path. What is the period (the time for one revolution)?

   c. Write the world line in the moving coordinate frame using the time in the moving frame as a parameter.

8. A particle of mass m moving relativistically with momentum p is projected at a second stationary particle also of mass m.

   a. Find the total 4-momentum (i.e., momentum-energy 4-vector) of the system.

   b. Find a coordinate frame in which the total 3-mom (i.e., 3 space-like components which are the momentum) is zero.

9. Two particles leave a collision point at 90° to the initial direction in the C.M. frame. If the velocity of the C.M. frame is c what are the angles in the lab frame? What is the magnitude of the momentum in the lab frame?

10. An event is seen in S_1 to be at the spacetime point $(10.0\,\mathrm{m}, 0, 0, tc)$, where $t = 1.0 \times 10^{-6}\,\mathrm{s}$. At what spacetime point is this event in S_2, which is moving at $0.6\,c$ (in the y-direction) with respect to S_1?

11. A particle of mass m_0, momentum $p = m_0c$, hits an unknown object in such a way that it is stopped. Write its initial and final momentum 4-vectors. Transform the difference of these 4-vectors to find a frame where the collision appears to be elastic (i.e., the energy is the same before as after so that the energy component of the 4-vector difference is zero). Give β for the frame.

Brief Answers:

1. $\begin{pmatrix} 1000 \\ 0 \end{pmatrix}$

2. $\begin{pmatrix} \gamma & -\frac{1}{2}\gamma \\ -\frac{1}{2}\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 1000\gamma \\ -\frac{1}{2}1000\gamma \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} \times 1000 \\ -\frac{1}{\sqrt{3}} \times 1000 \end{pmatrix}$

   which used:

$$\gamma = \frac{1}{[1-(1/4)]^{1/2}} = \frac{1}{(3/4)^{1/2}} = \frac{2}{\sqrt{3}}$$

3. a. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ b. $\begin{pmatrix} 0 \\ \frac{c}{v} L_2 \end{pmatrix}$

4. a. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ b. $\begin{pmatrix} L_1 \\ \frac{c}{v} L_1 \end{pmatrix}$

5. $$\begin{pmatrix} 0 \\ \dfrac{L_2}{\beta} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} L_1 \\ \dfrac{L_1}{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta\gamma L_1 + \dfrac{\gamma L_1}{\beta} \end{pmatrix}$$

$$\Longrightarrow L_2 = \gamma L_1 (1-\beta^2) = (1-\beta^2)^{1/2} L_1$$

6. $$\left[r\cos\left(2\pi \frac{t}{T}\right), r\sin\left(2\pi \frac{t}{T}\right), 0, ct \right]$$

7. a. Lorentz Transformation in the z-direction:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} r\cos(2\pi t/T) \\ r\sin(2\pi t/T) \\ 0 \\ ct \end{pmatrix}$$

$$= \begin{pmatrix} r\cos(2\pi t/T) \\ r\sin(2\pi t/T) \\ -v\gamma t \\ c\gamma t \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}$$

b. From answer to part (a): $t' = \gamma t$

$$x' = r\cos\left(\frac{2\pi t'}{\gamma T}\right)$$
$$y' = r\sin\left(\frac{2\pi t'}{\gamma T}\right)$$
$$z' = -vt'$$

Therefore the period is: $\gamma T = T'$.

c. $$\begin{pmatrix} r\cos(2\pi t'/T') \\ r\sin(2\pi t'/T') \\ -vt' \\ ct' \end{pmatrix}$$

8. a. $$\begin{pmatrix} p \\ E_1/c \end{pmatrix} + \begin{pmatrix} 0 \\ mc \end{pmatrix} = \begin{pmatrix} p \\ \left(m^2c^2 + p^2\right)^{1/2} + mc \end{pmatrix}$$

b. $$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} p \\ E/c \end{pmatrix} = \begin{pmatrix} \gamma(p - \beta E/c) \\ (-\beta p + E/c)\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \neq 0 \end{pmatrix}$$

where: $E = \left(m^2c^2 + p^2\right)^{1/2} + mc$

Therefore we obtain $\beta = pc/E$, which is different from the single particle formula. Note that this β is not the velocity for any particular particle since $E/c = mc + (m^2c^2 + p^2)^{1/2}$.

9. $$\begin{pmatrix} \gamma & 0 & +\beta\gamma \\ 0 & 1 & 0 \\ +\beta\gamma & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p \\ \left(m^2c^2 + p^2\right)^{1/2} \end{pmatrix} =$$

$$\begin{pmatrix} \beta\gamma\left(m^2c^2 + p^2\right)^{1/2} \\ p \\ \gamma\left(m^2c^2 + p^2\right)^{1/2} \end{pmatrix}$$

$$|p_L|^2 = p^2 + \frac{\beta^2}{1-\beta^2}\left(m^2c^2 + p^2\right)$$

$$\theta = \tan^{-1}\left(\frac{p(1-\beta^2)^{1/2}}{\beta\left(m^2c^2 + p^2\right)^{1/2}}\right)$$

10. $\beta = 0.6$; $\gamma = \dfrac{1}{(1-0.36)^{1/2}} = \dfrac{1}{0.8}$

The Lorentz transformation in the y-direction is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\beta\gamma \\ 0 & 0 & 1 & 0 \\ 0 & -\beta\gamma & 0 & \gamma \end{pmatrix}. \text{ The event is: } \begin{pmatrix} x_1 \\ 0 \\ 0 \\ ct_1 \end{pmatrix},$$

where: $x_1 = 10\,\text{m}$, $t_1 = 10^{-6}$ sec.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\beta\gamma \\ 0 & 0 & 1 & 0 \\ 0 & -\beta\gamma & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 0 \\ 0 \\ ct_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ -v\gamma t_1 \\ 0 \\ \gamma c t_1 \end{pmatrix}$$

$$= \begin{pmatrix} 10\,\text{m} \\ -\dfrac{0.6}{0.8}\left(3\times 10^8\right)\times 10^{-6}\,\text{m} \\ 0 \\ \left(\dfrac{1}{0.8}\times 10^{-6}\,\text{s}\right) c \end{pmatrix} = \begin{pmatrix} 10\,\text{m} \\ -225\,\text{m} \\ 0 \\ c\,1.25\times 10^{-6}\,\text{s} \end{pmatrix}$$

11. $\begin{pmatrix} m_0 c \\ \sqrt{2} m_0 c \end{pmatrix} - \begin{pmatrix} 0 \\ m_0 c \end{pmatrix} = \begin{pmatrix} m_0 c \\ (\sqrt{2}-1) m_0 c \end{pmatrix}$

$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} m_0 c \\ \left(\sqrt{2}-1\right) m_0 c \end{pmatrix} = \begin{pmatrix} \gamma m_0 c \left(1 - \beta\left(\sqrt{2}-1\right)\right) \\ \gamma m_0 c \left(-\beta + \sqrt{2} - 1\right) \end{pmatrix}$

$\Longrightarrow \beta = \sqrt{2} - 1$

MODEL EXAM

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2. Show that energy and momentum of a particle are the components of a four-vector. Let the particle have velocity u_x, u_y, u_z in one Lorentz frame and u'_x, u'_y, u'_z in a second frame moving with velocity v along the x-axis with respect to the first frame. The velocity addition law is:

$$u'_x = \frac{u_x - \beta c}{(1 - \frac{u_x}{c}\beta)}$$

$$u'_y = \frac{1}{\gamma} \frac{u_y}{(1 - \frac{u_x}{c}\beta)}$$

$$u'_z = \frac{1}{\gamma} \frac{u_z}{(1 - \frac{u_x}{c}\beta)}$$

$$\frac{1}{\sqrt{1 - (\frac{u'}{c})^2}} = \gamma \frac{1 - \frac{u_x}{c}\beta}{\sqrt{1 - (\frac{u}{c})^2}}$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$.

3. An event is seen in S_1 to be at the spacetime point ($10.0\,\mathrm{m}$, 0, 0, tc), where $t = 1.0 \times 10^{-6}\,\mathrm{s}$. At what spacetime point is this event in S_2, which is moving at $0.6\,c$ (in the y-direction) with respect to S_1?

4. A particle of mass m_0, momentum $p = m_0 c$, hits an unknown object in such a way that it is stopped. Write its initial and final momentum 4-vectors. Transform the difference of these 4-vectors to find a frame where the collision appears to be elastic (i.e., the energy is the same before as after so that the energy component of the 4-vector difference is zero). Give β for the frame.

Brief Answers:

1. See this module's *text*.
2. See this module's *text*.
3. See this module's *Problem Supplement*, problem 10.
4. See this module's *Problem Supplement*, problem 11.