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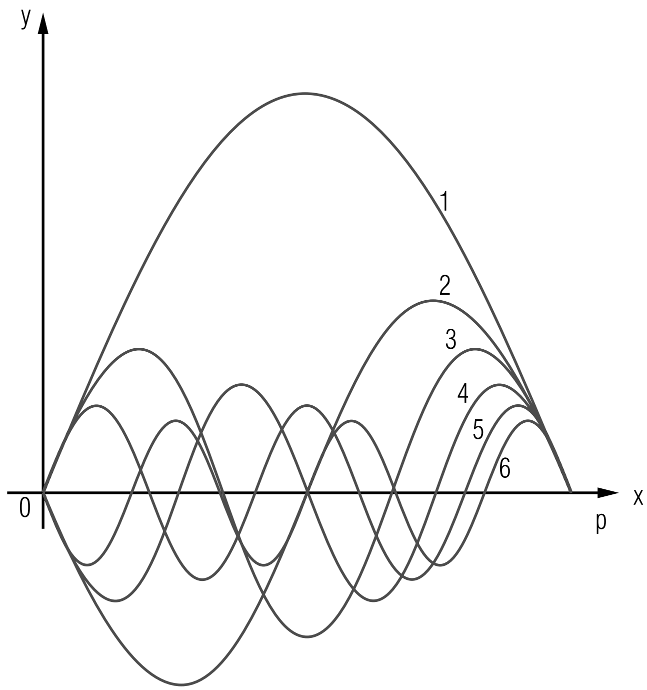
MISN-0-50

FOURIER SERIES AND INTEGRALS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FOURIER SERIES AND INTEGRALS

by

E. H. Carlson, Michigan State University

Title: **Fourier Series and Integrals**

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Length: 1 hr; 24 pages

Input Skills:

1. Compute definite and indefinite integrals of simple functions, including sine and cosine functions (MISN-0-1).
2. Understand the definite integral as an area (MISN-0-1).
3. Be familiar with the possibility of expansion of a function in a power series (MISN-0-4).

Output Skills (Knowledge):

K2. State sufficient conditions for the existence of the Fourier transform of a function.

Output Skills (Rule Application):

R1. Estimate the sizes of the Fourier coefficients by inspection of $f(x)$, considering its overlap with sine and cosine functions and noting discontinuities, cusps, peaks, wiggles in $f(x)$ of size ℓ, and symmetry.

R2. Compute the sine and cosine Fourier transform of a given $f(x)$.

R3. Sketch, by inspection, the Fourier transform of a given $f(x)$.

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E. H. Carlson, Michigan State University

1. Introduction

Suppose you have a function $f(x)$ defined in an interval

$$-L/2 < x < L/2$$

on the x-axis, as in Fig. 1.

You are probably familiar with the notion that, if $f(x)$ is sufficiently well behaved, you can expand it in a power series:

$$f(x) = a_0 + a_1 x + a_2 x + \ldots,$$

a Taylor Series. It is also possible to expand it in a series of sine and cosine functions:

$$f(x) = a_0 + a_1 \cos\left(2\pi \frac{x}{L}\right) + a_2 \cos\left(2\pi \frac{2x}{L}\right) + \ldots +$$

$$b_1 \sin\left(2\pi \frac{x}{L}\right) + a_2 \sin\left(2\pi \frac{2x}{L}\right) + \ldots,$$

a Fourier Series. Here are the advantages of using a Fourier expansion:

1. Many problems, such as those involving waves and oscillations, are particularly simple when expressed this way. That is because they generally have some periodicity, some interval of x over which $f(x)$ repeats.

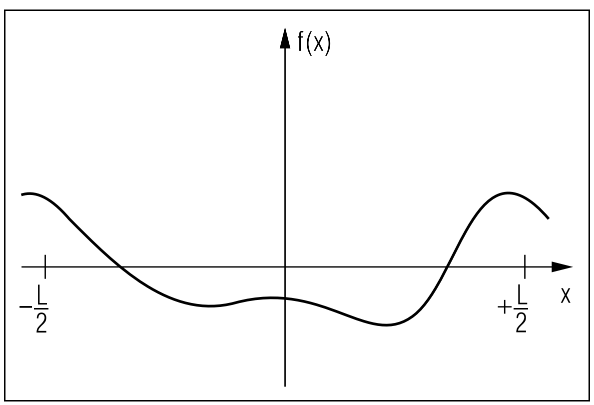

Figure 1. Some function which we wish to represent with an appropriate series.

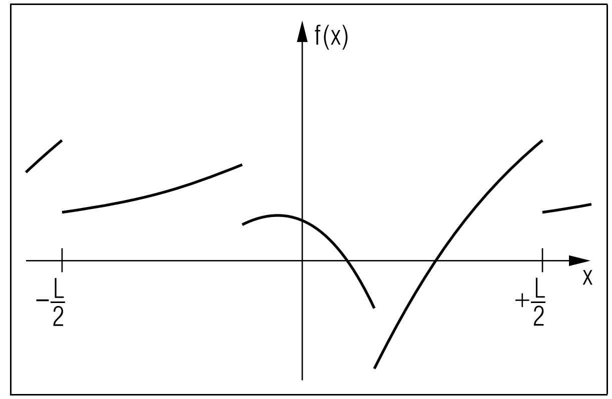

Figure 2. A function $f(x)$ having three discontinuities, counting the two at $(\pm L/2)$ as one (they are the same).

2. The criteria that $f(x)$ must satisfy, in order that the series converge, are not very stringent. It is sufficient that, in the interval $-L/2 < x < L/2$, $f(x)$ is finite and has a finite number of maxima and minima. It may even have a finite number of discontinuities. These are called Dirichlet's conditions.

3. If $f(x)$ is not periodic in x, we can still use the general idea by letting $L \to \infty$, thereby obtaining the Fourier Integral representation:

$$f(x) = \int_{-\infty}^{\infty} [A(k)\cos(kx) + B(k)\sin(kx)]\, dk \tag{1}$$

2. The Fourier Series

2a. The Coefficient Equations. We will discuss the relationship between $f(x)$ and the coefficients a_k, b_k, leaving derivations and proofs to a mathematics text. The coefficients are defined by the integrals:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, dx \\ a_k &= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(2\pi kx/L)\, dx \\ b_k &= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin(2\pi kx/L)\, dx. \end{aligned} \tag{2}$$

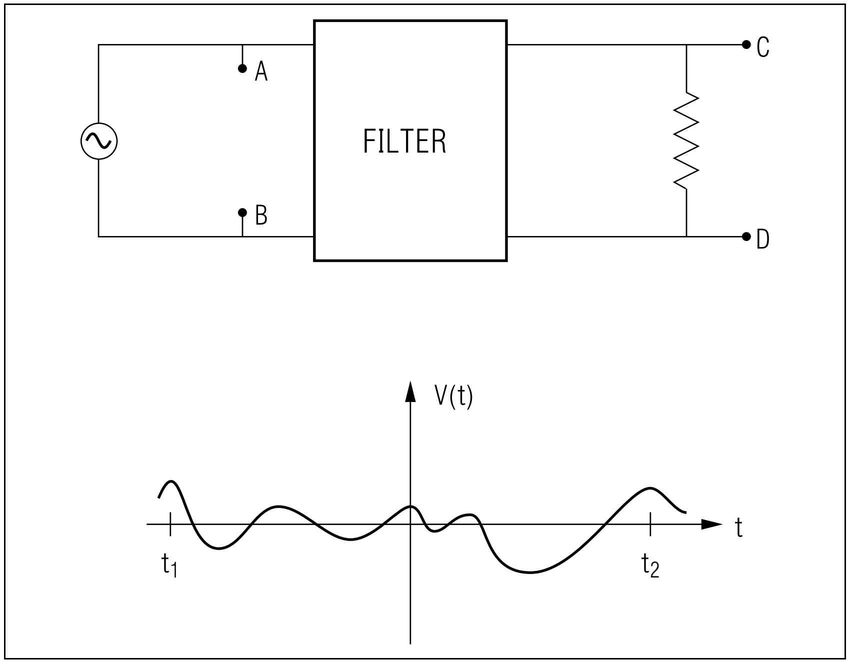

Figure 3. A voltage wave that is input to a filter.

2b. An Example. Fourier Series are useful not only as a computational tool, for which use Eqns. (2) must be evaluated, but even more so as a conceptual tool which simplifies the description of $f(x)$ for many applications. An example is shown in Fig. 3. Here the generator produces a repetitive time-varying wave form whose voltage across points A, B is $V(t)$. The voltage across points C, D will be most clearly expressed when the Fourier Series for $V(t)$ is known. Write:

$$V(t) = V_0 \sum_{k=1}^{\infty} (a_k \cos \omega_0 kt + b_k \sin \omega_0 kt).$$

If the filter passes only frequencies above some frequency ω_1 then only voltage Fourier components for which $k > \omega_1/\omega_0 0$ will appear at points C, D. One will usually begin the analysis of such a physical problem by inspecting $f(x)$ and seeing which coefficients a_k, b_k are large and which are small or zero. This gives insight into the solution, guidance in calculating the series coefficients, and a check on possible gross errors in the solution.

2c. Partial Sum and Formal Definition. Suppose we keep only the first $2n + 1$ terms in the Fourier Series, as we certainly would do in any numerical calculation of the coefficients. This defines the "partial sum" ϕ:

$$\phi_n(x) = a_0 + \sum_{k=1}^{n} [a_n \cos(2\pi kx/L) + b_n \sin(2\pi kx/L)]. \qquad (3)$$

The series obtained as $n \to \infty$ defines the Fourier Series if a_k and b_k are calculated using the Fourier coefficient equations given below.

3d. Non-Periodic but Localized Functions. For many physical situations $f(x)$ will be "localized," i.e., $f(x) \to 0$ as $x \to \pm\infty$. For example, $f(x)$ may represent a pulse or a wave train of finite dimensions. Then one can simply pick an interval L so large that it contains essentially all of $f(x)$, i.e., so that $f(x) \approx 0$ for $|x| > L/2$. Then $f(x)$ can be represented by a Fourier Series inside the interval but not outside the interval. Outside the interval we abandon the Fourier Series and simply set $f(x)$ equal to zero.

▷ Show that the resulting Fourier Series will not correctly represent $f(x)$ outside the interval.

2e. Estimating the Coefficients. Here is how professionals estimate the coefficients to see which are important, which are marginal, and which are negligible:

1. a_0 is just the average value of $f(x)$ over the interval.

2. Each a_k, b_k is proportional to the "overlap" of the corresponding cosine or sine function with $f(x)$. That is, when $f(x)$ and $\cos 2\pi kx/L$ are large and positive in the same places, negative or zero in the same places etc., then a_k will be large and positive. (What shape must $f(x)$ have for a_k to be large and negative?) The "overlap" idea is central to our method for roughly evaluating the integrals of Eqns. (2), and some particular cases will be discussed in the next few numbered remarks.

3. Low values of k contribute (through a_k and b_k) to the overall, broad outline of $f(x)$, while smaller scale structures (wiggles, peaks, etc.) that occupy a length $\ell < L$ on the x-axis require contributions from sine and cosine functions whose wavelength λ is near ℓ in size ($\lambda = L/k$).

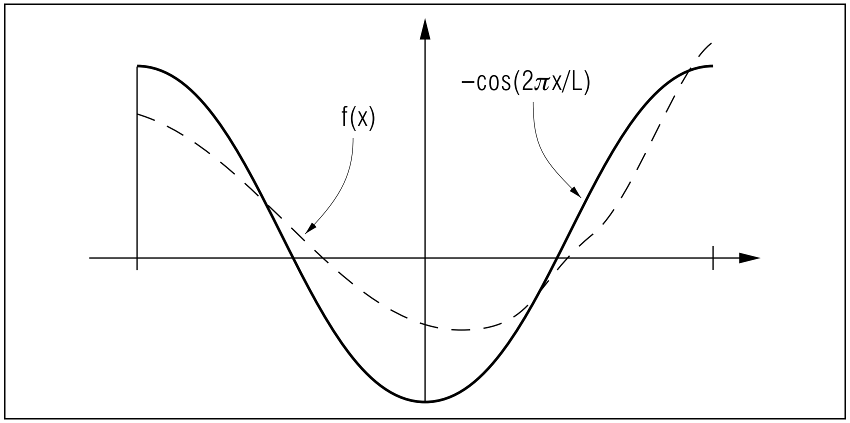

Figure 4. The overlap is large between $f(x)$ and $\cos(2\pi x/L)$. What about the overlap of $f(x)$ with $\sin(2\pi x/L)$?

4. If the size of the smallest structure in $f(x)$ is ℓ, then a_k, b_k fall off in size rapidly as k becomes much larger than L/k. The reason can be seen from Fig. 6.

5. The polynomials ϕ_n are periodic in x, so $\phi_n(x \pm mL) = \phi_n(x)$, where m is an integer. Thus $f(x)$ (which we did not necessarily define outside of $-L/2 < x < L/2$) is treated as being periodic. This may introduce a discontinuity or cusp at the points $x = +L/2$.

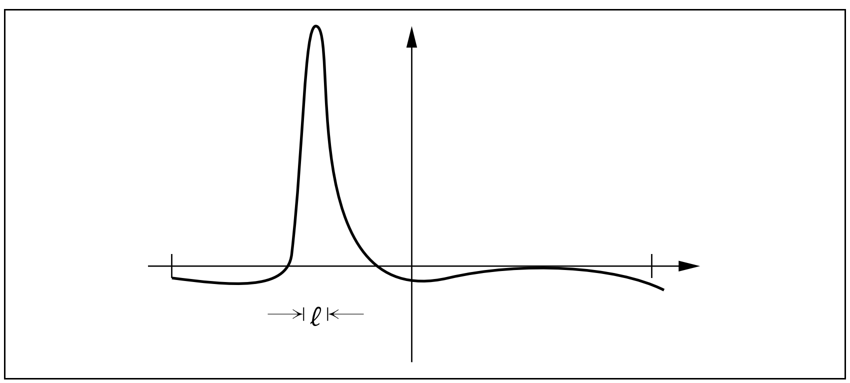

Figure 5. Coefficients a_k, b_k with $k = L/\ell$ will be relatively large.

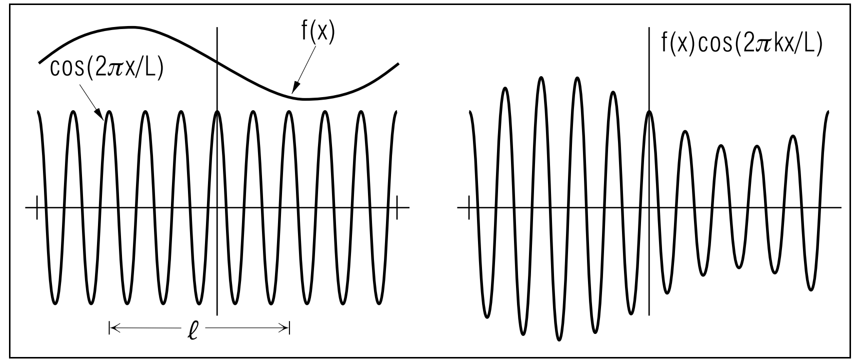

Figure 6. The integral defining a_k has many positive lobes that are nearly cancelled by the adjacent negative one of nearly the same size; so the whole integral is small.

6. Discontinuities in $f(x)$ are "structure" whose characteristic dimension ℓ is zero. They introduce the requirement that $a_k \propto 1/k$, $b_k \propto 1/k$ as $k \to \infty$, and likewise, "cusps" (where df/dx is discontinuous) give a_k, $b_k \propto 1/k^2$.

7. The symmetry of $f(x)$ may simplify the series. A function $f(x)$ is called "even" if $f(x) = f(-x)$ and "odd" if $f(x) = -f(-x)$. (What are the symmetries of $\sin x$, of $\cos x$?) If $f(x)$ is even, only a_k's will be non zero, if $f(x)$ is odd, only b_k's will be non zero. (Why?)

8. The partial sum $\phi_n(x)$ is a least squares fit to $f(x)$. The error in representing $f(x)$ by $\phi_n(x)$ is $e_n(x) = f(x) - \phi_n(x)$. Of all the possible methods of choosing the coefficients a_k, b_k, that given by

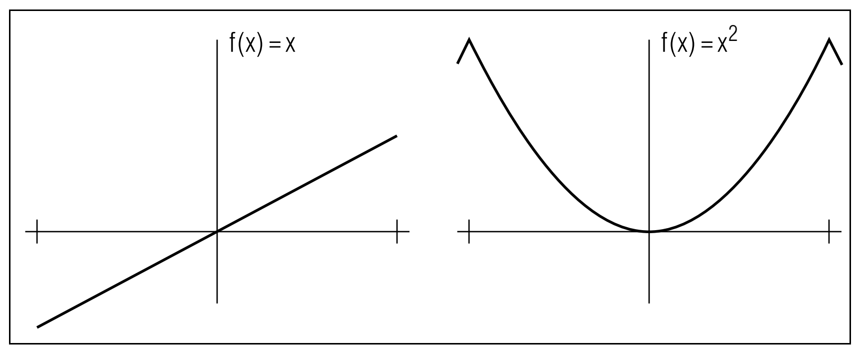

Figure 7.

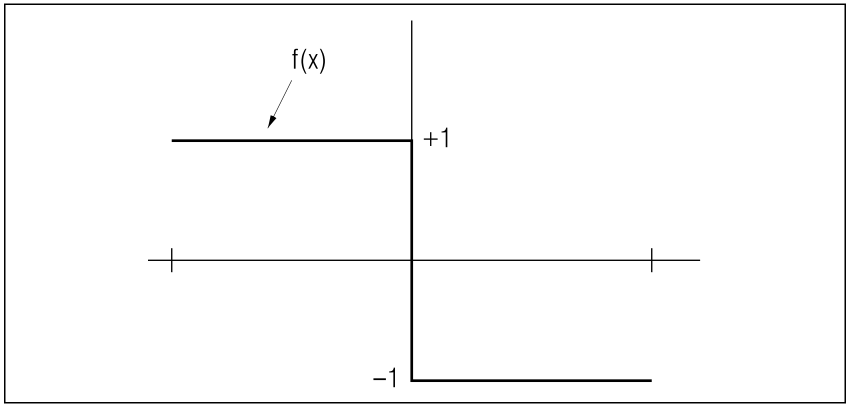

Figure 8. Fourier transform of a step function. $f(x) = -4/\pi \sum_{k=1,3,5,...} (1/k)\sin(2\pi kx/L)$. Can you explain why the k-even terms are missing? Hint: sketch-in $\sin(4\pi x/L)$. Explain why there is a minus sign in front.

Eqns. (2) minimizes the integral $I = \int |e_n(x)|^2\, dx$.

9. From the fact that Eqns. (2) do not contain n, we see that when we approximate $f(x)$ by $\phi_n(x)$ and determine the coefficients a_k, b_k, $k < n$, and then decide to make a better approximation $\phi_p(x)$, $p > n$, the coefficients a_k, b_k for $k < n$ already determined will not change.

10. A single wave at wavelength $\lambda = L/\ell$ cannot, of course, form the peak. There must be many other waves of wave lengths near $\lambda =$

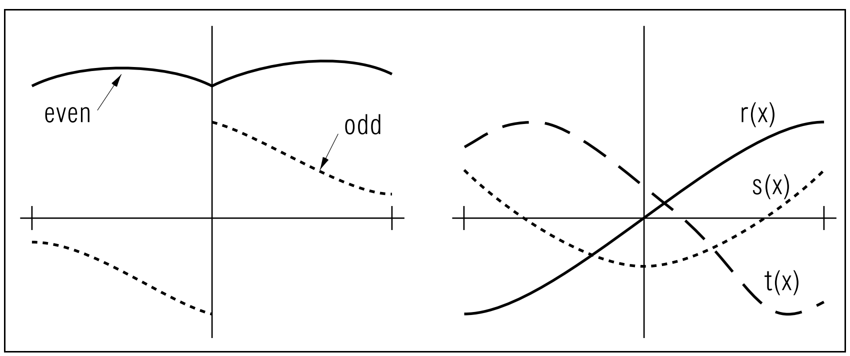

Figure 9. Which of r, s, t is even, odd, neither? Can one function be both even and odd?

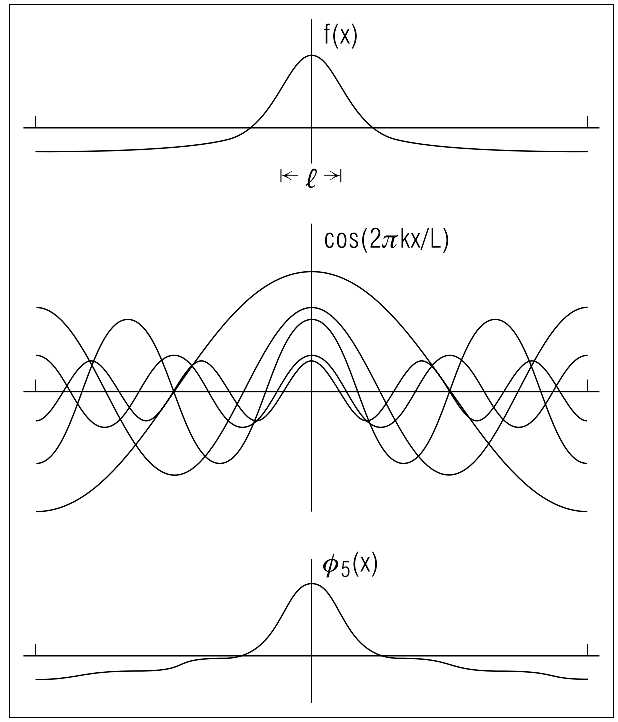

Figure 10. The cosine functions are plotted with 4-fold vertical exaggeration for clarity.

L/ℓ which add to each other at the position of the peak and cancel each other elsewhere.

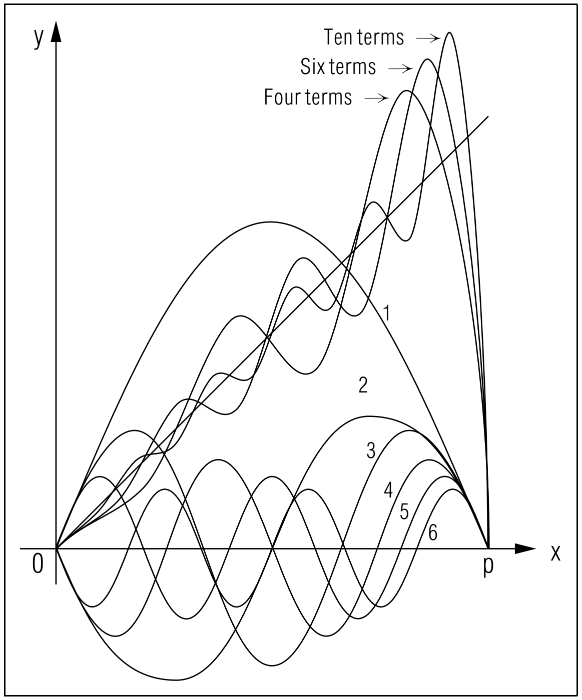

Figure 11. Representation of $y(x) = x, -\pi \leq \pi$, by $\phi_n(x)$ for $n = 4, 6, 10$ (only the region where $x > 0$ is shown). Here $b_k = (2/k) \cdot (-1)^k$, $a_k = 0$.

3. The Fourier Integral

3a. Series vs. Integral. If a function $f(x)$ is not periodic or is not restricted to a finite interval of x, the function cannot be expanded in a Fourier Series and one must turn to the Fourier Integral. This is equivalent to letting the period or the localization interval go to infinity so the sum in the Fourier Series becomes an integral.

3b. Transition: Series to Integral. Here we will make the transition from the Fourier Series to the Fourier Integral.

For simplicity of derivation, let $f(x)$ be an even function. Assume a periodicity or locality of length L and suppose we have already obtained a set of Fourier Series coefficients a_k where $k = 0, 1, 2, \ldots$. We can write the arguments of the sine and cosine functions as:

$$2\pi\frac{kx}{L} = 2\pi\frac{x}{L/k} = 2\pi\frac{x}{\lambda_k}$$

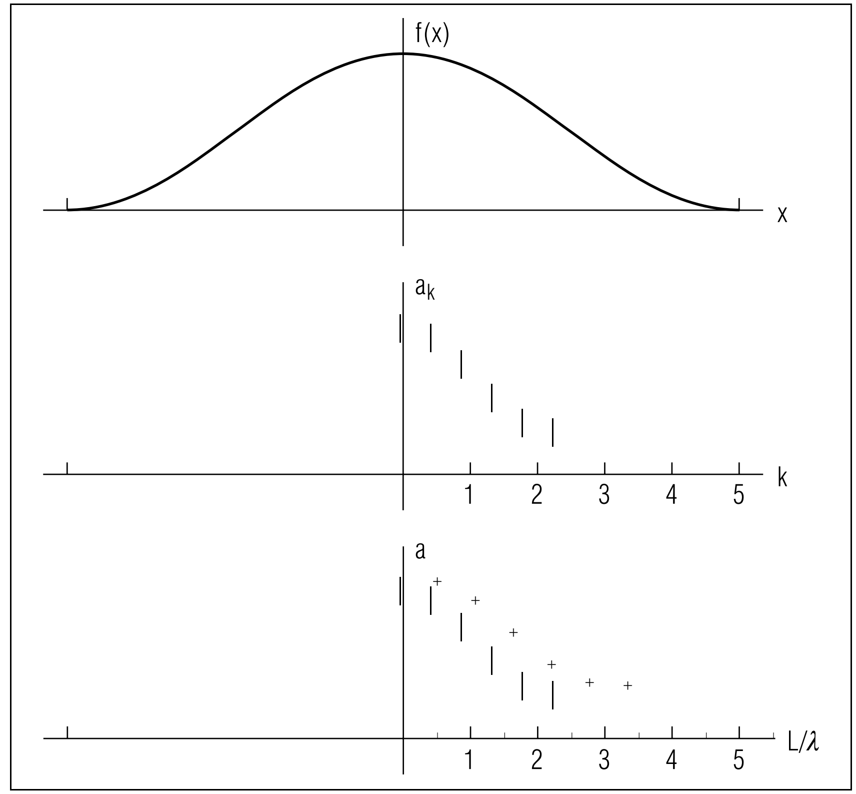

Figure 12. Fourier coefficients added by going to the larger interval are marked with crosses.

where $\lambda_k = L/k$ is obviously the wavelength of the k^{th} wave in the series. Note that exactly k complete waves fit into the periodicity distance L.

Now suppose we need to make the the interval twice as large, so the interval is $-L \leq x \leq L$, and we recalculate the a_k's. We will be using the wavelengths $\lambda_k = 2L/k = 2L$, L, $2L/3$, $L/2$, ... so all the coefficients we calculated before will still be here but with different k subscripts. That means we can use the original graph and not have to rename anything if we label things by their *wavelength*, instead of by the k integer. Writing the abscissa as $\lambda = L/k$ instead of k, we have Fig. 11.

As we let the interval L grow larger and larger, the scale of the graph will change and the points labeled by integer values of k will get ever more closely spaced. As $L \to \infty$ the points go toward becoming a continuum.

3c. The Continuum Case. In the continuum case we want to describe the continuum by some parameter that looks like our series-case $k = L/\lambda$ but which does not involve L. It is traditional to use the quantity called the "wave number" k defined by:

$$k = 2\pi/\lambda.$$

The Fourier coefficients a_k, b_k, with integer k, now become the "Fourier amplitudes" $A(k)$, $B(k)$, with a continuous dimensional k.

▷ What are k's dimensions?

Skipping further details on the transition from a sum to an integral, we write the equations equivalent to Eq. (3) and Eq. (2).

If $f(x)$ obeys Dirichlet's conditions in every finite interval, no matter how large, and if, in addition,

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

is finite, then

$$f(x) = \int_{-\infty}^{\infty} [A(k)\cos(kx) + B(k)\sin(kx)]\, dk \tag{4}$$

$$\begin{aligned} A(k) &= \int_{-\infty}^{\infty} f(x)\cos(kx)\, dx \\ B(k) &= \int_{-\infty}^{\infty} f(x)\sin(kx)\, dx. \end{aligned} \tag{5}$$

Note that k can take on negative values. This feature will be important when traveling waves, rather than standing waves, are used. However, for our purposes, $A(k)$ is even and $B(k)$ is odd and we will only plot the portion of each that has k positive.

3d. Eyeballing the Amplitudes. Just as for Fourier Series, the Fourier analysis of a function $f(x)$ into waves of various amplitudes and wavelengths can clarify its physical properties, and it is often sufficient to get a rough idea of the shape of $A(k)$ and $B(k)$ by "eyeballing" the function $f(x)$. Most of the ideas presented in the discussion of Fourier Series are still valid, including symmetry, overlap, structure size verses wavelength, etc.

Let us consider the examples in Fig. 13. For small k (long wavelength) $cos(kx) \approx 1$ and so near the origin, $A(k)$ is constant and equal to the area

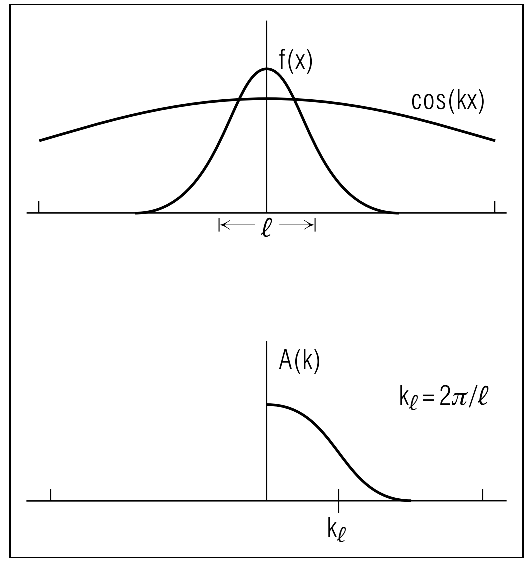

Figure 13.

under $f(x)$ divided by 2π. For k large [R smaller than any structure in $f(x)$], $A(k)$ approaches zero from cancellation of the + and − lobes of the integral (see Fig. 6). The region in which $A(k)$ drops off rapidly is near $\lambda = 2\pi/k\ell \approx \ell$, the size of the major structure in $f(x)$.

▷ For the examples in Figs. 14-16, see if you can justify exactly the form of $A(k)$ and $B(k)$.

The most elegant and useful form of the Fourier integral comes when we use the notation of complex numbers. Then

$$e^{ikx} = \cos(kx) + i\sin(kx), \qquad i = \sqrt{-1},$$

and we write for the real (or complex) valued function $f(x)$ of the real variable x:

$$f(x) = \int_{-\infty}^{\infty} G(k)e^{ikx}\, dk \tag{6}$$

where

$$G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dk \tag{7}$$

is the *Fourier transform of* $f(x)$ and is generally a complex valued function. If $f(x)$ is real, then $G(k)$ is related to the Fourier amplitudes of Eq. (5) by:

$$A(k) = ReG(k)$$

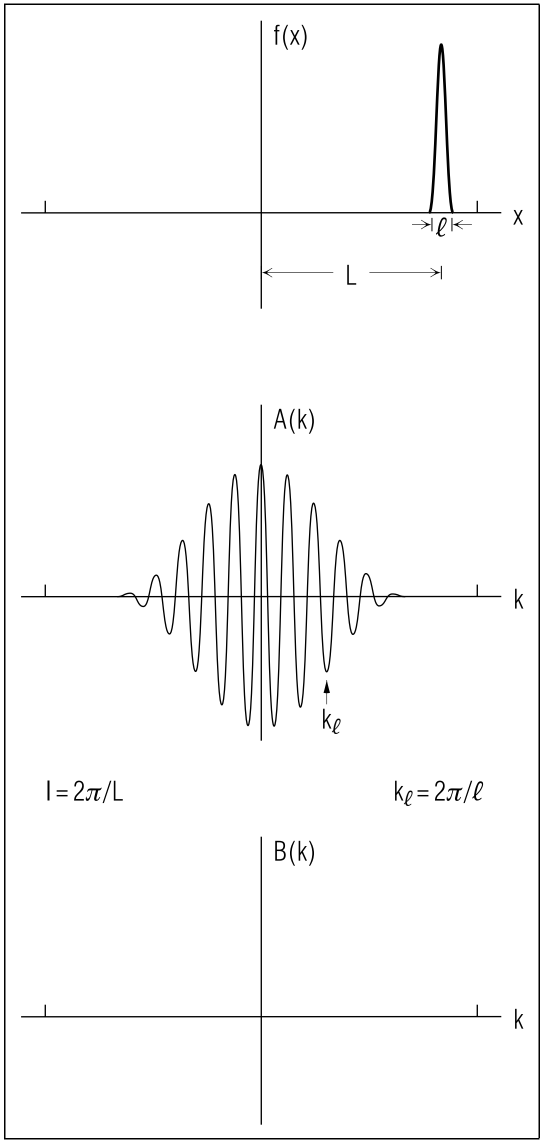

Figure 14.

$$B(k) = -ImG(k).$$

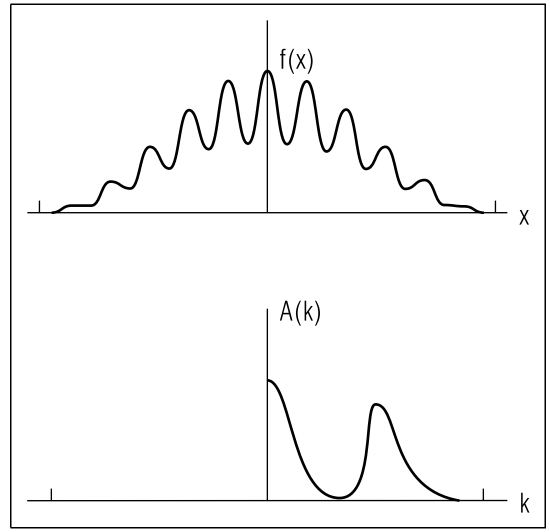

Figure 15.

Acknowledgments

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A. Some Indefinite Integrals

$$\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$\int x^2 \sin ax\, dx = \frac{2x}{a^2} \sin ax - \frac{x^2a^2 - 2}{a^3} \cos ax$$

$$\int x^3 \sin ax\, dx = \frac{3x^2a^2 - 6}{a^4} \sin ax - \frac{a^2x^3 - 6x}{a^3} \cos ax$$

$$\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos ax\, dx = \frac{2x}{a^2} \cos ax + \frac{a^2x^2 - 2}{a^3} \sin ax$$

$$\int x^3 \cos ax\, dx = \frac{3x^2a^2 - 6}{a^4} \cos ax + \frac{a^2x^3 - 6x}{a^3} \sin ax$$

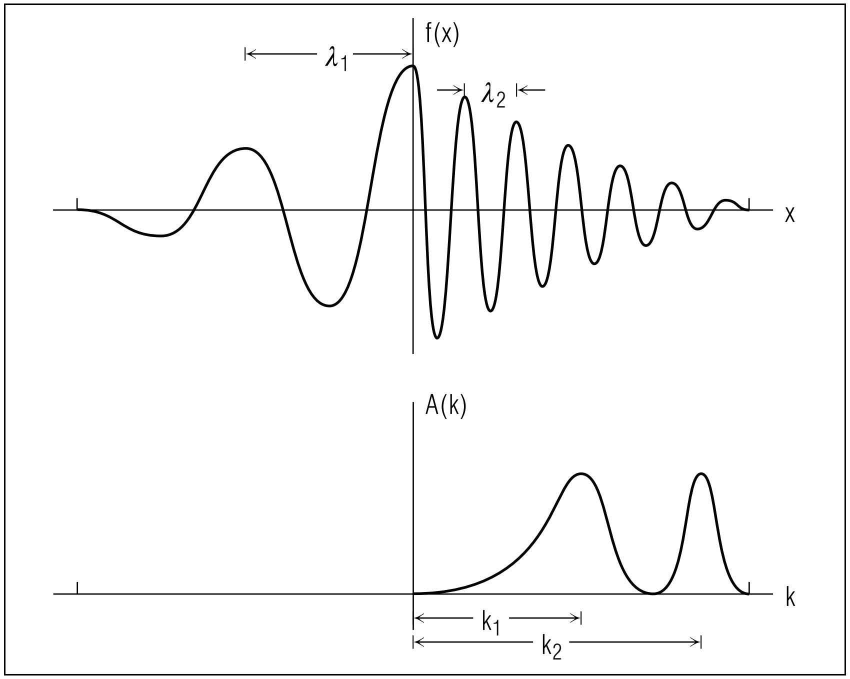


Figure 16.

$\int e^{ax} \sin px \, dx = e^{ax} \left(\dfrac{a \sin px - p \cos px}{a^2 + p^2} \right)$

$\int e^{ax} \cos px \, dx = e^{ax} \left(\dfrac{a \cos px + p \sin px}{a^2 + p^2} \right)$

B. A Definite Integral

$$\int_0^\infty e^{-a^2/x^2} \cos bx \, dx = \frac{\sqrt{\pi}}{2a} e^{(-b^2/4a^2)} \,, \qquad (ab \neq 0)$$

PROBLEM SUPPLEMENT

1. For each function listed below, sketch the function, apply symmetry conditions to see if any set of coefficients are zero, consider structure and overall shape to predict which coefficients may be large, consider the rate at which coefficients approach zero as $k \to \infty$ and then use Eqs. (2) to evaluate the coefficients. Each function is defined in $-\pi \leq x \leq \pi$.

   (a) $f(x) = 1$ for $|x| \leq \pi/2$
   $\quad\;\; = 0$ elsewhere

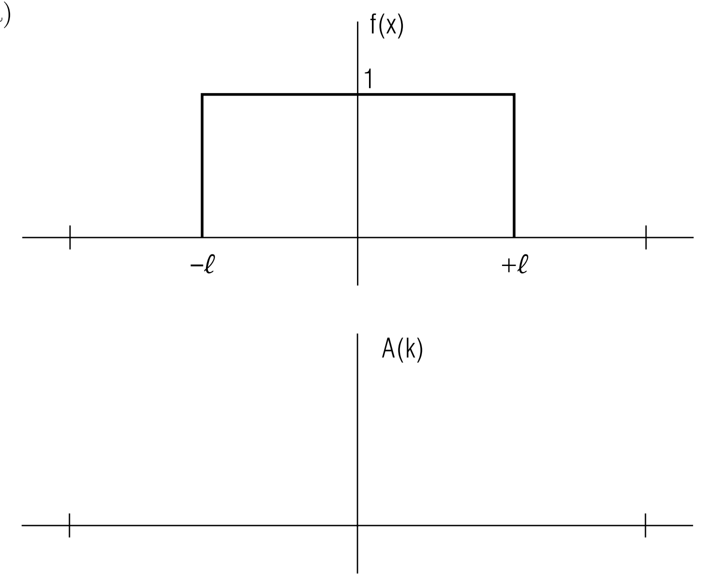
   (b) $f(x) = x$ for $|x| \leq \pi/2$
   $\quad\;\; = 0$ elsewhere

   (c) $f(x) = x$ for $0 \leq x \leq \pi$
   $\quad\;\; = 0$ elsewhere

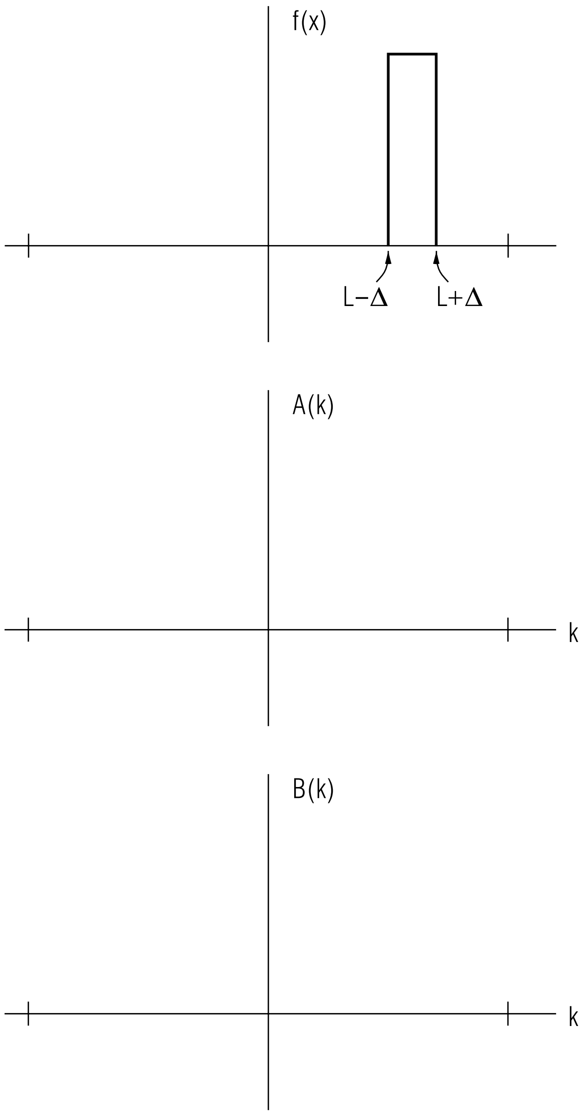
   (d) $f(x) = x^2$ in the interval.

2. For each function below, sketch $A(k)$, $B(k)$ by inspection of $f(x)$, then compute $A(k)$ and $B(k)$ and compare.

   (a)



(b)



(c)