PHYSNET

MISN-0-480

INTEGRAL CALCULUS FOR VECTORS

Math

Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

INTEGRAL CALCULUS FOR VECTORS

by

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1. Introduction 1

2. Procedures 2

Acknowledgments 2

Title: **Integral Calculus for Vectors**

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Input Skills:

1. Vocabulary: line, surface, and volume integrals.
2. Unknown: assume (MISN-0-479).

Output Skills (Knowledge):

K1. Write down Gauss's divergence theorem.

K2. Write down Stokes's theorem.

K3. Recognize some of the alternate form of Gauss's theorem (such as Green's theorem).

K4. Recognize some of the alternate forms of Stokes's theorem.

K5. Prove the various forms of Gauss's theorem, starting with: $\int_S \vec{V} \cdot d\vec{\sigma} = \int_V \vec{\nabla} \cdot \vec{V}\, d\tau$.

K6. Prove the various forms of Stokes's theorem, beginning with: $\oint \vec{V} \cdot d\vec{\lambda} = \int_s (\vec{\nabla} \times \vec{V}) \cdot d\vec{\sigma}$.

Output Skills (Rule Application):

R1. Evaluate integrals involving derivatives of scalar and vector fields using the various integral theorems.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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New authors, reviewers and field testers are welcome.

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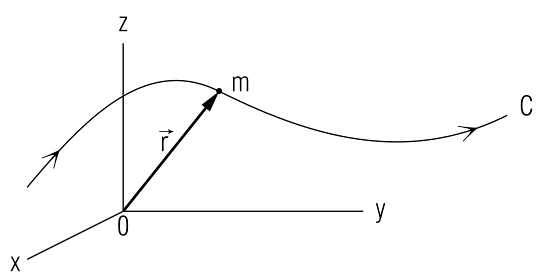
1. Introduction

In this Unit, various integral theorems involving vectors are developed. Many physical quantities can be expressed in terms of vector integrals. For example, work (W) associated with a force ($\vec{F}$) being exerted on an object traversing a curve (C) is actually a line integral,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

where C is a space curve being traversed by a particle of mass m located at position $\vec{r}$ in C as shown in the diagram below:



O represents the origin of the rectangular (x, y, z) coordinate system.

Another example is electric flux ϕ across a surface S in an electric field $\vec{E}$. The electric flux is then given by this surface integral:

$$\phi = \int_S \vec{E} \cdot d\vec{\sigma}$$

where $d\vec{\sigma} = \hat{n}\, dS$ and $\hat{n}$ is the positive unit normal to the surface S. When the surface S encloses a volume V, Gauss's Divergence Theorem, which will be studied in this unit, allows us to rewrite this as:

$$\phi = \int_V \vec{\nabla} \cdot \vec{E}\, d\tau$$

where $d\tau$ is the element of volume. The use of Maxwell's equations allows us to relate the electric flux to the total charge Q contained in the volume

V by

$$\phi = \frac{1}{\epsilon_0} Q\,.$$

There are many other examples of the physical applications of the material in this unit.

2. Procedures

1. Read in Arfken, section 1.11, and Spiegel, page 154, and Supplementary Problem 6.30. Write down Gauss's Divergence Theorem. Be certain to memorize the conditions on the vector fields and any derivatives of vector fields for the theorem to be valid.

2. Read in Arfken, section 1.12, and Spiegel, page 154 and Supplementary Problem 6.34. Write down Stokes's Theorem. Be certain to memorize the conditions on the vector fields and and derivatives of vector fields for the theorem to be valid.

3. Write down the derivation of eqs. 1.108, 1.109, 1.112, and 1.113 of Arfken.

4. Solve these problems in Arfken: 1.11.1, 1.11.3, 1.12.1, 1.12.2. These problems deal with the numerical evaluation of integrals involving vectors by using Gauss's Divergence Theorem and Stokes's Theorem.

Note: The purpose of this unit is to learn how to use Gauss's and Stokes's Theorems *without* a detailed numerical computation of the line, surface, and volume integrals. You should already have mastered line, surface, and volume integrals.

Acknowledgments

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