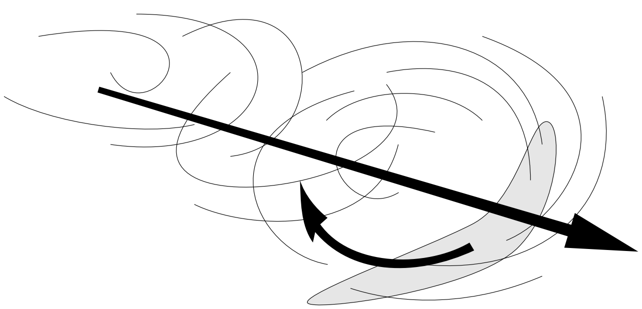
PHYSNET

MISN-0-43

TRANSLATIONAL & ROTATIONAL MOTION OF A RIGID BODY



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TRANSLATIONAL & ROTATIONAL MOTION OF A RIGID BODY

by

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Input Skills:

1. Calculate the kinetic energy loss in an inelastic collision (MISN-0-21).
2. Calculate the kinetic energy of a rigid rotating object (MISN-0-36).

Output Skills (Problem Solving):

S1. Apply the following theorems to determine completely the motion of a rigid body in the case when the torque on the body is along one of the principal axes: (a) Relative to an inertial reference system the time rate of change of the center of mass momentum of a rigid body is equal to the net external force acting on the rigid body; (b) Relative to any point in an inertial reference system, the time derivative of the angular momentum vector of a rigid body is equal to the net external torque acting on the rigid body; and (c) The total kinetic energy of a rigid body relative to an inertial reference system is the kinetic energy of the center of mass of that body relative to the inertial system, plus the kinetic energy of the body relative to the center of mass.

Post-Options:

1. "Ideal Collisions Between a Frictionful Sphere and a Flat Surface: The Superball" (MISN-0-53).
2. "Euler's Equations: The Tennis Racket Theorem" (MISN-0-57).

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1. Introduction

1a. General Description of the Motion of a System of Particles. The general description of the motion of a system of particles consists of specifying the motion of the center of mass (CM) of the system (as if the whole mass of the system were located at the center of mass) plus the description of the motion of the particles of the system relative to the center of mass. For a rigid body the particles of the system are constrained to move such that the relative separation of all pairs of particles remains unchanged. Theorems about the motion of rigid bodies will be demonstrated as they apply to a simple rigid system.

1b. Theorems on the Motion of Rigid Bodies. When a set of forces acting upon a rigid body are such that they combine to produce an external torque which is directed along one of the principal axes of the body (see Fig. 1), the following theorems may be applied to determine completely the motion of the rigid body:

a. Relative to an inertial reference system the time rate of change of the center of mass momentum of a rigid body is equal to the net external force acting on the rigid body:

$$\frac{d\vec{P}_{CM}}{dt} = \vec{F}_{ext}.$$

a′. If there is no net external force on a rigid body, the center of mass momentum is constant, and hence the center of mass velocity is constant in magnitude and direction: $\vec{P}_{CM}$ = constant vector, $\vec{V}_{CM}$ = constant vector.

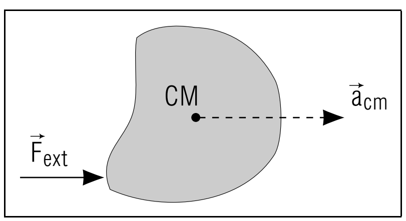

Figure 1. Torque-producing forces acting on a body.

b. Relative to any point in an inertial reference system, the time derivative of the angular momentum vector of a rigid body is equal to the net external torque acting on that body relative to the same point:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}.$$

b′. With no external torque on the body the angular momentum vector relative to any point in an inertial reference system is constant.

c. The total kinetic energy of a rigid body relative to an inertial reference system is the kinetic energy of the center of mass of that body relative to the inertial system, plus the kinetic energy of the body relative to the center of mass. (For a rigid body this latter is the kinetic energy of the rotation of the body about the center of mass.

2. Application of Rigid Body Theorems to a Simple System

2a. Description of the Motion of a Rigid Body. The complete description of the motion of a rigid body consists of the description of the translational motion of its center of mass as if it were a point mass, plus the description of its rotational motion about an axis through its center of mass.[1] The net external force on the system will determine the former, while the net external torque will determine the latter.

2b. Example: Collision of a Point Mass with a Dumbbell. As an illustration of the theorems listed in Sect. 1b, consider the following example:

In far outer space, far from the influence of any appreciable gravitational force, a rigid body consisting of two point masses, of mass M and $4M$, at the ends of a massless rod of length d is at rest relative to an inertial reference frame.[2] Another mass, M, makes a collision approach

[1]The axis is not necessarily fixed in its direction in space. In cases when it is not, the motion of the direction of the axis is necessary to complete the description. We will not consider such cases here. See, however, "Torque and Angular Momentum in Circular Motion" (MISN-0-34) and "Euler's Equations: The Tennis Racket Theorem" (MISN-O-57).

[2]An inertial reference frame is a frame in which Newton's First Law of Motion is true.

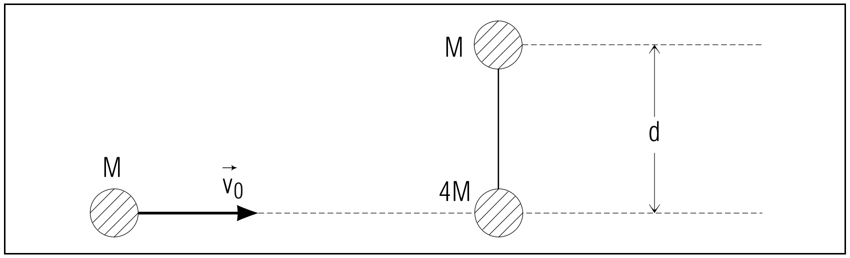

Figure 2. A mass M with initial speed $\vec{V}_0$ approaches to make a head on collision with a mass $4M$, initially at rest. The mass $4M$ is attached via a rigid massless rod to another mass M with the orientation of the rod such that it is perpendicular to the line of flight of the moving mass.

toward this rigid body, moving with a velocity $\vec{V}_o$ toward a head on collision with the $4M$ mass directed perpendicularly to the rod connecting the rigid body masses (See Fig. 2).

2c. Mass and Dumbbell: Translational Motion. *Problem*: To demonstrate that the center of mass of the three mass system moves with the same velocity before and after the collision.

Answer: First, the center of mass must be located, then its velocity before the collision occurs must be determined. The center of mass of the rigid body is located one-fifth of the distance from the $4M$ mass along the line connecting the two masses (at point A in Fig. 3). The center of mass of the combination of this mass ($5M$) and the incident mass M is located along the line joining these masses and one-sixth of the distance from the $5M$ mass to the incident mass (at point B). Figure 2 shows this location at the instant when the incident mass is distance D from the mass $4M$.

Problem: How high above the line joining the incident mass with the struck mass is the center of mass?

Answer: From the similar triangles OPA and $OB'B$ we see that BB' is five-sixths of AP so that the length of BB' is $(d/6)$.

Problem: Where will the center of mass be when the incident mass strikes the $4M$ mass?

Answer: It will be at B'', a distance $(d/6)$ above point P, the location of the mass $4M$. With B and B'' both the same distance above the line OP, it is clear that the line BB'' is parallel to the line OP. Thus, while

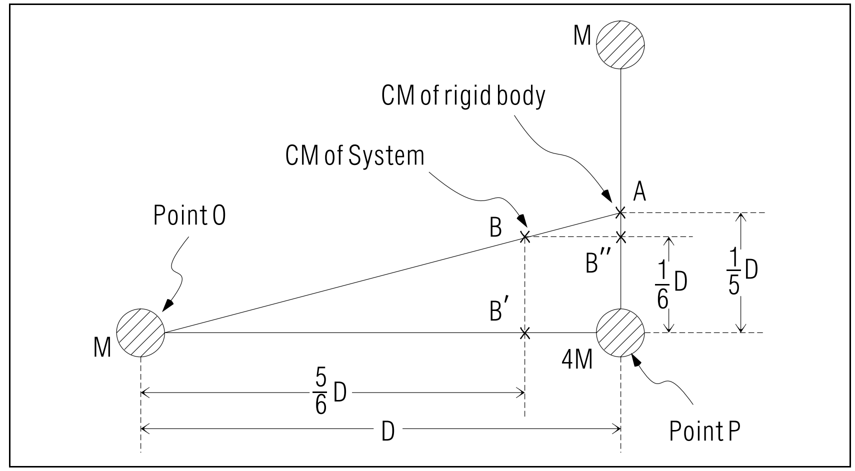

Figure 3. A geometrical determination of the location of the center of mass of the system described in Fig. 2.

the incident mass moved distance D, the center of mass moved from B to B'' along a line parallel to line OP. Line BB'' is of length $(D/6)$, hence the center of mass velocity must be one-sixth of the incident velocity and parallel to the incident velocity:

$$\vec{V}_{CM} = \frac{1}{6}\vec{V}_0. \tag{1}$$

Problem: What is the CM velocity after the collision?

Answer: It must be the same because there are no external forces on the three mass system. (The incident mass and the struck mass exert forces on each other but these are internal forces.) Because :

$$\frac{d\vec{P}_{CM}}{dt} = \vec{F}_{ext} = 0, \tag{2}$$

$$\vec{P}_{CM} = \text{ constant vector.} \tag{3}$$

To demonstrate this, consider the case where the incident mass sticks to the mass $4M$ upon colliding. Before the collision, as observed in an inertial frame at rest with respect to the rigid body, the only momentum is $M\vec{V}_0$. After the collision the momentum is that of the new rigid body ($5M$ at one end, M at the other). The momentum is thus $6M\vec{V}_{CM}$ where

$\vec{V}_{CM}$ is the velocity of the center of mass after the collision. Equating these two (because there are no external forces) we get:

$$\vec{V}_{CM} = \frac{1}{6}\vec{V}_0, \tag{4}$$

exactly what it was before the collision.

2d. Mass and Dumbbell: Rotational Motion. *Problem*: Is this the only motion of the combined system?

Answer: Obviously not. The system spins around as it moves. (If the rigid rod were struck at the center of mass of the system, no spinning would occur.) To describe completely the motion we must also include a description of the rotation of the system. Rotational motion is determined by external torques on the system and how they affect the angular momentum of the system:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}, \tag{5}$$

where $\vec{L}$ and $\vec{\tau}_{ext}$ are both defined with respect to the same point in some inertial reference frame. With our system (consisting of the three masses) there is no net external torque. Therefore the angular momentum, evaluated relative to any point in an inertial frame, is constant:

$$\vec{L} = \text{ constant vector.} \tag{6}$$

It is convenient to take the point relative to which the torque is evaluated as the center of mass of the system.

Problem: First, is this a point in an inertial reference frame?

Answer: If the reference frame in which the rigid rod (before collision) is at rest is an inertial reference frame, then so is a frame which is attached to the center of mass and moving with it. That's because the center of mass reference frame is moving with constant velocity with respect to the reference frame which is at rest with respect to the rigid rod. (In frames of reference which don't accelerate with respect to each other, observers see the same force or effect of a force.)

Problem: Relative to this point, what is the angular momentum of the system before the collision takes place?

Answer: Relative to this point the incident mass is moving to the right with velocity $(5/6)\vec{V}_0$ (recall that the center of mass itself is moving to the right with velocity $(1/6)\vec{V}_0$). The masses on the ends of the rigid rod

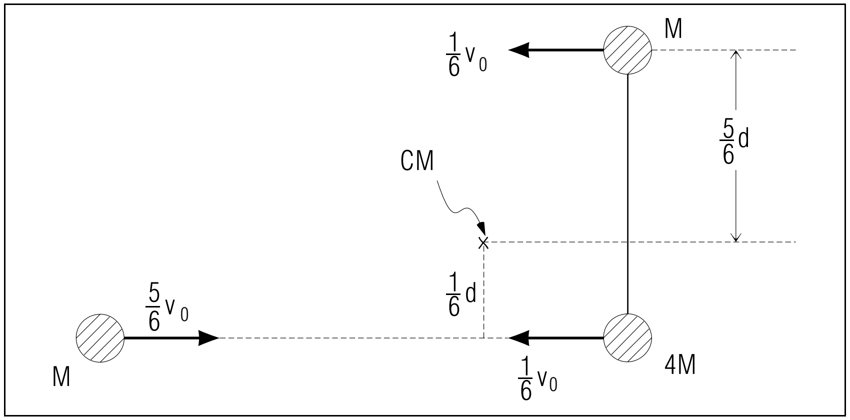

Figure 4. The motion of the system described in Fig. 2 (before collision) as observed in a reference frame in which the center of mass is at rest.

are thus both moving to the left with a speed $(1/6)\vec{V}_0$ or with velocity $-(1/6)\vec{V}_0$. Hence, viewed at rest with respect to the center of mass, the motion observed is as shown in Fig. 4.

The total angular momentum of this system as seen relative to the center of mass is the vector sum of the individual angular momenta. Each $\vec{L} = \vec{r} \times m\vec{v}$, where $\vec{r}$ is the vector from the center of mass to the instantaneous location of the mass, and $\vec{v}$ is the velocity of that mass. However, the magnitude of this cross product is the magnitude of the momentum times the "lever arm."[3] Its direction may be evaluated separately using the right hand rule. The individual angular momenta are determined as indicated in Table 1.

[3] See "Force and Torque" (MISN-0-5)

Table 1. Determination of the individual angular momenta relative to the center of mass of the system for each of the three point masses of the system shown in Fig. 2.

Object	Mass	Speed	Lever Arm	Magn. of Ang. Mom.	Direct. of Ang. Mom.
incident mass	M	$\frac{5}{6}V_0$	$\frac{1}{6}d$	$\frac{5}{36}MV_0d$	up, out of page
mass on rod	$4M$	$\frac{1}{6}V_0$	$\frac{1}{6}d$	$\frac{4}{36}MV_0d$	down, into page
mass on rod	M	$\frac{1}{6}V_0$	$\frac{5}{6}d$	$\frac{5}{36}MV_0d$	up, out of page

The resultant angular momentum of the system (before collision) is thus:

$$\frac{1}{6}MV_0d \text{ directed up, out of page.} \tag{7}$$

This is also the value of the angular momentum of the rod after the incident mass struck and attached to the $4M$ mass. Hence the newly formed rigid body (M and $5M$) at two ends of a rigid rod of length d rotates counter-clockwise about the center of mass with an angular momentum whose magnitude is $(d/6)MV_0$.

Problem: What is the angular velocity of rotation of this rigid body?

Answer: For a rigid body rotating about a principal axis $\vec{L} = I\vec{\omega}$ and $\vec{\omega}$ is in the same direction as $\vec{L}$. In this case, up out of page: the rotation is counter-clockwise. I, the moment of inertia, from its definition is the sum of the "mass times distance squared" contributions for each mass relative to the axis parallel to $\vec{L}$ through the center of mass. It is, therefore,

$$I = M\left(\frac{5}{6}d\right)^2 + 5M\left(\frac{1}{6}d\right)^2 = \frac{5}{6}Md^2. \tag{8}$$

We thus find: $\vec{\omega} = V_0/(5d)$. Consequently, the complete description of the motion of the system after impact is as follows:

i. The center of mass moves with constant velocity $(1/6)\vec{V}_0$ in a straight line parallel to the direction of the velocity of the incident mass.

ii. The rod rotates counter-clockwise about an axis through the center of mass (perpendicular to the plane of the paper) with a constant angular velocity of $V_0/(5d)$

2e. Mass and Dumbbell: Kinetic Energy. *Problem*: What is the kinetic energy of the system?

Answer: Again the answer depends upon the reference frame with respect to which this kinematical quantity is to be observed. Hence, making the question more specific:

Problem: What is the kinetic energy of the system relative to a system which is at rest with respect to the rigid rod before the collision?

Answer: The kinetic energy before the collision is that of the incident mass:

$$\frac{1}{2}MV_0^2.$$

After the collision it is the kinetic energy of the center of mass as if all the mass were concentrated there: $(1/2)6M\dfrac{V_0}{6}^2$ plus the kinetic energy of all the parts of the system relative to the center of mass. For a rigid body this is $(1/2)I\vec{\omega}^2$. For this object it is:

$$\frac{1}{2}I\vec{\omega}^2 = \frac{1}{60}MV_0^2. \tag{9}$$

The total kinetic energy is thus:

$$E_k(\text{total}) = \frac{1}{2}MV_0^2 + \frac{1}{60}MV_0^2 = \frac{1}{10}MV_0^2. \tag{10}$$

(5 times as much kinetic energy is in translational motion as there is in rotational motion. If the incident mass had struck the center of mass, all the kinetic energy would have been in translational motion.)

Problem: What is the "Q" of this collision?

Answer: According to its definition, Q is the change in the kinetic energy that occurs as a result of the reaction. If Q is positive, kinetic energy has been gained as a result of the reaction or collision. For this collision:

$$Q = -\frac{4}{5}\left(\frac{1}{2}MV_0^2\right). \tag{11}$$

As a result of the collision 80% of the original kinetic energy is lost. (The collision is obviously inelastic[4]).

[4]See "Potential Energy, Conservative Forces, The Law of Conservation of Energy" (MISN-0-21).

Acknowledgments

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PROBLEM SUPPLEMENT

1. For the system shown in Fig. 1 of the textual material, determine the velocity of the center of mass of the system before the collision. Show that it is parallel to the velocity of the incident mass.

2. Determine the velocity of the center of mass of the system of Problem 1 after the collision, assuming the projectile mass sticks to the mass $4M$.

3. Relative to the center of mass of this above system, determine the velocity of each of the three component masses before the collision.

4. Determine the total angular momentum relative to the center of mass of the above mentioned system before the collision.

5. Determine the total angular momentum relative to the stationary mass M (on the rod) before the collision.

6. Assuming conservation of angular momentum, calculate the angular velocity of the system relative to the center of mass after the collision.

7. Assuming conservation of angular momentum, calculate the instantaneous angular velocity of the system after the collision relative to the location of the initially stationary mass M. (This will be the angular velocity with which that upper mass M sees the other masses begin to rotate around it.)

8. Determine the total kinetic energy, relative to a frame at rest with respect to the rod, of the system before the collision occurs.

9. Determine the total kinetic energy relative to this same frame (as in 8 above) of the system after the collision.

10. Determine the gain or loss of kinetic energy as a result of the collision.

Brief Answers:

1. Equation (1).
2. Equation (2).
3. See Fig. 3.
4. Equation (7).
5. MV_0d, directed up, out of page.
6. $\omega = V_0/(5d)$ counter-clockwise as viewed looking at the plane of the paper.
7. $\omega = V_0/(5d)$.
8. $\frac{1}{2}MV_0$.
9. Equation (11).

MODEL EXAM

1. For the same problem as was illustrated in the *text*, Sect. 2, find the kinetic energy of the system before the collision [K], the kinetic energy after the collision [C] and the Q of the collision [F], all as observed in a reference frame attached to the center of mass.

2. In far outer space free from any outside forces and observed in an inertial frame, there is at rest a uniform rod of length L and mass M. A point mass, also of mass M approaches the rod with a velocity $\vec{V}_0$ directed perpendicular to the long axis of the rod. The mass strikes the rod at a point that is a distance d away from the center of the rod, toweard the end of the rod. After the collision, the incident mass continues along the same straight line in its original direction but with a reduced velocity, $\vec{V}_0/2$.

   a. Determine the location of the center of mass of the system at an instant when the incident mass is a distance D from the rod. [L] Determine the velocity of the center of mass at this instant. [A]

   b. Determine the velocity of the center of mass of the rod after the collision, as observed in the same frame where the rod was initially at rest. [I]

   c. Determine the velocity of the center of mass of the rod relative to the center of mass of the system after the collision. [J]

   d. Determine the angular momentum of the system, before the collision, relative to the center of mass of the rod. [B]

   e. Determine the angular velocity of the rod about its own center of mass after the collision. [E]

   f. Determine the kinetic energy of the system before the collision, as observed in the original inertial frame. [H]

   g. Determine the kinetic energy of the system after the collision, as observed in this same frame. [M]

   h. Determine the Q of this reaction. [G]

   i. Assuming an elastic collision, determine the value of d in terms of L. [D]

Brief Answers:

A. $\vec{V}_0/2$

B. $(MV_0 d)$, directed up, out of page, as seen in a sketch where d is below center.

C. $MV_0^2/60$

D. $d = L/\sqrt{6}$

E. $6V_0 d/L^2$

F. $-(4/10)MV_0^2$

G. $(1/2)MV_0^2\left(\dfrac{3d^2}{L^2} - \dfrac{1}{2}\right)$

H. $MV_0^2/2$

I. $(\vec{V}_0/2)$, same direction as original velocity of point mass.

J. Zero

K. $(5/12)MV_0^2$

L. Along a line joining the center of the rod to the incident mass and half-way between the two.

M. $(1/2)MV_0^2\left(\dfrac{1}{2} + \dfrac{3d^2}{L^2}\right)$