PHYSNET

MISN-0-380

THE DIRAC DELTA FUNCTION

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE DIRAC DELTA FUNCTION

by

R. Spital

Title: **The Dirac Delta Function**

Author: R. Spital, Dept. of Physics, Illinois State Univ

Version: 2/25/2000 Evaluation: Stage B0

Length: 1 hr; 8 pages

Input Skills:

1. Integrate these functions: Gaussian, exponential, trigonometric (MISN-0-1).
2. Perform simple double integrals (MISN-0-6).

Output Skills (Knowledge):

K1. Write down equations 11 on page 35 of *Elementary Quantum Mechanics*, by Saxon and define the term "Fourier transform."

K2. Starting from the equations of objective (1), derive equation 12 on page 36 of Saxon and the corresponding representation of the delta function (equation 13) below it.

K3. List the three most important properties of the delta function.

K4. Derive equations 14, 15, 16, 17 on page 37 of Saxon.

Output Skills (Rule Application):

R1. Use the delta function in order to trivially solve double integrals that contain one of the variables only in the argument of an imaginary exponential and only linearly there.

External Resources (Required):

1. D. S. Saxon, *Elementary Quantum Mechanics*, Holden-Day, Inc. (1968).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

THE DIRAC DELTA FUNCTION

by

R. Spital

1. Introduction

In Quantum Mechanics one sometimes needs the wave functions in coordinate space, sometimes in momentum space. The wave function for a wave with definite momentum is a plane wave in coordinate space and a Dirac delta function in momentum space. In fact, Dirac invented the delta function for just this purpose. Although it is a strange quantity, not really a function in the mathematical sense, engineers and scientists usually think of it as a function and use like a function. Because they always integrate a delta function before identifying the result with a physical quantity, their use of the delta function has been shown to be rigorously correct mathematically. Engineers and scientists have found the delta function to have powerful uses in classical mechanics, in electricity and magnetism, and in the study of moving fluids.

2. Procedures

1. a. Use the readings from Saxon for this section. These pages are available in the PA library. Ask for them as "the readings for CBI Unit 383."

   b. Read section 3 of Chapter 3 through the end of page 35. Memorize equations 11.

2. Follow the steps at the top of page 36 of Saxon to reach equations 12 and 13. Then, without the use of books or notes, write down equations 11 and derive equations 12 and 13.

3. The three most important properties of the Dirac delta function are:

   a. $\delta(x) = 0$ for $x \neq 0$

   b. $\delta(x) = \infty$ for $x = 0$

   c. $\int_a^b f(x)\delta(x-c)dx = f(c)$, provided c lies between a and b; otherwise the integral is 0.

   Clearly the delta function is not a proper function in the mathematical sense. It can only be given meaning inside integrals. One might think

   of it as a very tall spike centered at the origin in the limit as the height of the spike goes to infinity with the area under the spike held fixed at 1.

4. Equation 14 is a special case of equation 12. To derive the other results change the variable of integration in equation 12 to be the argument of the delta function; then do the integration. For example:

   Prove: $\delta(x) = \delta(-x)$

   Proof: Let $u = -x$. Then $\int_{-\infty}^{\infty} f(x)\delta(-x)dx = -\int_{\infty}^{-\infty} f(u)\delta(u)du = \int_{-\infty}^{\infty} f(u)\delta(u)du = f(0)$.

   Hence, $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(x)\delta(-x)dx$, where f(x) is almost any function.

   (Actually in order to apply condition (c) of procedure 3, f must have a Taylor series expansion about the origin; i. e. f must be a "well-behaved" function). This implies $\delta(x) = \delta(-x)$.

5. Follow the proof of equation 19 on page 37 of Saxon, making sure you understand each step. When you arrive at the top of page 38, put in the delta function and finish the integration. Be sure you understand the interchange of the order of integration so as to produce a delta function and make the integral trivial. Then let $k = 0$ and $f_1(x) = f_2^*(x)$ in equation 19 to produce equation 20.

6. For Output Skill R1, think about how you would solve an integral like this:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(p-q)x} f(p)\, dx\, dp\,.$$

   Answer: $2\pi f(q)$. Now do it several times, substituting different functions for $f(x)$. For example, be able to work this through properly:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(p-q)x} p^2\, dx\, dp = 2\pi q^2\,.$$

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.