PHYSNET

MISN-0-353

WAVES IN TWO DIMENSIONS

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

WAVES IN TWO DIMENSIONS

by

Robert Ehrlich

Title: **Waves in Two Dimensions**

Author: R. Ehrlich, Physics Dept., George Mason Univ.

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Input Skills:

1. Vocabulary: Doppler effect, shock wave, Cerenkov radiation (MISN-0-204); coherent sources of waves, interference (MISN-0-230); Fourier synthesized function & coefficients (MISN-0-352).
2. Use a nested loop two levels deep to calculate a given function over a grid of points in the x-y plane (MISN-0-348).
3. Use the technique of intensity scaling to produce intensity plots of a given function (MISN-0-348).

Output Skills (Knowledge):

K1. Vocabulary: aspect ratio, beat frequency, beats, image enhancement, overprinting technique, two-dimensional Fourier synthesis.

Output Skills (Project):

P1. Enter and run a computer program to produce wave intensity plots for a point source moving at various speeds. Make measurements on the output and confirm the formulas for the Doppler effect and for shock waves using your measurement.
P2. Enter and run a computer program to produce wave intensity plots for two interfering point sources using a range of values for the source separation. Make measurements on the output and confirm the equations for the intensity maxima and minima using your measurements.
P3. Modify the program in P2 to introduce an intrinsic phase shift ϕ in one of the two interfering sources. Run the program using several values of ϕ and report the effect on the pattern.
P4. Modify the program in P2 to make the two sources incoherent. Run the program and observe the lack of interference maxima and minima.

External Resources (Required):

1. A computer with FORTRAN or BASIC.

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1. Overview

In this module the computer is used to graphically illustrate and analyze a number of situations involving waves in two dimensions. A program accompanying the module can generate an intensity plot of the resultant wave in each of these cases:

1. A single stationary point source

2. A single moving point source (Doppler effect and shock waves)

3. Two coherent point sources (interference).

The computer produces the intensity plots using the technique of intensity scaling.[1] In addition, the FORTRAN version of the program, and the first of the two C++ versions, use "overprinting" in which several lines of characters are printed on top of each other in order to introduce various gradations in darkness.[2] The overprinting feature is not available on some printers. Accordingly, the BASIC version of the program, and the second C++ version, do not require overprinting.

2. Point Sources of Waves

2a. Point Sources of Waves in a Plane. An example of a wave in two dimensions is the surface wave produced when a small object is dropped in calm water. The amplitude of the wave, which in this case is the vertical displacement of the water level from its equilibrium position, can be expressed as some function of two spatial coordinates x, y, z, and the time t. If the object bobs up and down in simple harmonic motion, it produces a sinusoidal wave which expands outward in all directions on the surface of the water (the xy-plane). As in the case of any transverse wave, the oscillatory motion of any point in the medium is perpendicular to the direction of propagation of the wave at that point.

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[1] See "Advanced Printer Graphics and Scaling" (MISN-0-348).

[2] The printer that you use should be moderately fast. Otherwise it will take too much time to produce each plot.
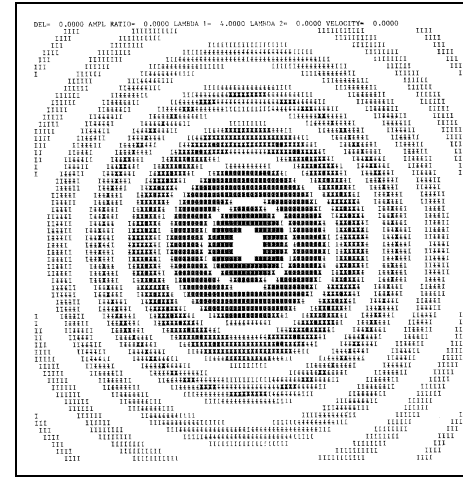
Figure 1. Computer generated intensity pattern for a point source of waves.

2b. Expression for the Wave Amplitude. If we were dealing with a plane wave travelling along the $+x$ direction we could write the following expression for the wave amplitude as a function of x and t:

$$A(x,t) = A_o \sin \left[\frac{2\pi}{\lambda}(x - vt) \right] \tag{1}$$

Where λ is the wavelength and v is the wave speed. Our interest, however, will be primarily in waves emanating from a point. There are two necessary modifications of Eq. (1) in this case:

a. The variable x must be replaced by the variable $r = (x^2 + y^2)^1/2$ which is the distance from the source, and

b. the constant A_o must be replaced by S/r where S is the strength of the source of waves.

With these two changes we find that the amplitude of the wave at any point (x, y) and time t is given by

$$A(x,y,t) = \frac{S}{r} \sin \left[\frac{2\pi}{\lambda}(r - vt) \right] \tag{2}$$

At any particular instant of time, say t = 0, the wave is represented by the function

$$A(x,y) = \frac{S}{r} \sin \left[\frac{2\pi}{\lambda} r \right] \tag{3}$$
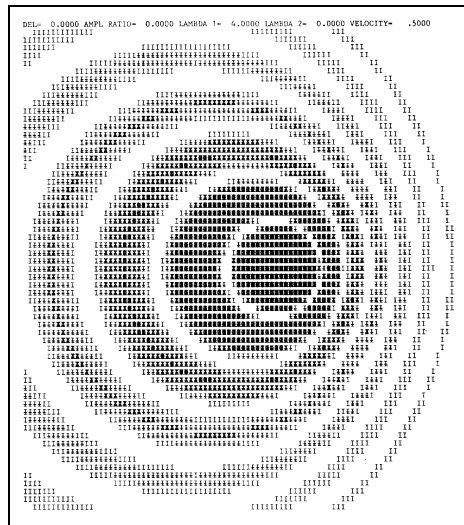
Figure 2. Computer generated intensity pattern for a moving point source of waves.

This wave amplitude can be represented by the computer-generated plot shown in Fig. 1 in which the darkness of the pattern at each point is an indication of the intensity, or square of the wave amplitude, at that point. (The procedure for doing this will be described). Note that as required by Eq. (3), the wave amplitude is zero on concentric circles of radii $r = n\lambda/2$, $n = 0, 1, 2, ...$, so that the distance between every other zero-amplitude circle is one wavelength. The alternating regions of positive and negative amplitude are indistinguishable in Fig. 1 since it is the intensity, not the amplitude, that is represented. The figure does indicate both the sinusoidal oscillations as a function of r and the more gradual inverse-square dependence of intensity on r, in accordance with the form of Eq. (3). A wave front may be defined as the set of all points on the wave having the same phase. This means that the argument of the sine in Eq. (3) is the same for all points on a wave front. For a point source at rest, the wave fronts must therefore form a set of concentric circles spaced one wavelength apart, expanding outward at the wave speed v. Both Fig. 1 and Eq. (3) apply only to the case of a point source which is at rest as far as motion in the xy-plane is concerned.

2c. Waves from a Moving Source. If the source is not at rest the wave fronts must still be circular because the waves expand outward at the same velocity in all directions. However, the circles are not concentric, since adjacent circular wave fronts correspond to waves emitted from the

source at different times, the center of each circle being the position of the source at the time that wave front was emitted from the source. For a source traveling at constant velocity, the center of each wave front is displaced by a constant amount from the one before. The computer-generated plot of Fig. 2, which shows a snapshot of the wave pattern for a uniformly moving source, illustrates this.

3. The Doppler Effect

3a. Introduction. The change in frequency which occurs when a wave source is in motion with respect to an observer is known as the "Doppler effect". The frequency detected by the observer is higher (lower) than that for a stationary source, when the source moves toward (away from) the observer. The Doppler effect also occurs when the observer, rather than the source, is moving, although this case will not be included in our discussion.

3b. Cause of the Doppler Effect. The cause of the Doppler effect can be easily understood with the aid of Fig. 2. Recall that the frequency, according to an observer, is the number of wave fronts reaching him per unit time. When a source approaches an observer (observer to the right of the source in Fig. 2), the spacing between wave fronts is less than what it would be if the source were at rest. Similarly, if a source recedes from an observer, the spacing between wave fronts is greater than for a source at rest. The wave fronts reach the observer at the same velocity, whether the source is at rest or in motion. Therefore, the waves must have a higher (lower) frequency for a source which approaches (recedes from) an observer than for a source at rest.

3c. Simplest Case: Head-On Approach. In the simplest case, the source approaches an observer head-on with constant velocity. For this case, the observer detects a constant higher frequency, up to the instant the source passes him; after this instant, he detects a constant lower frequency. The frequency he detects as a function of the position of the source therefore looks like the "step function" curve (a) in Fig. 3. This curve is consistent with the wave pattern in Fig. 2. To the right of the source, the wave front spacing is about 1/2 the specified wavelength; to the left of the source, the spacing is about 3/2 times the wavelength. Since the observed frequency is inversely proportional to the wave front spacing, the two levels of the step function in Fig. 3 correspond to frequencies which are respectively 2 and 2/3 times the frequency of the source at rest. These
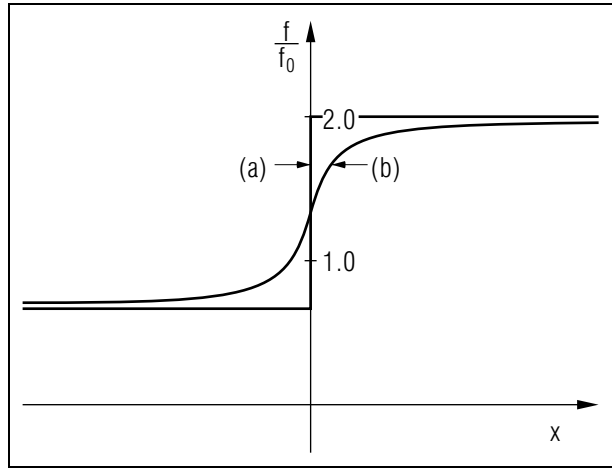
Figure 3. Frequency ratio f/f_0 versus position x for a source of waves moving at half the wave speed. Curve (a) is for head-on motion, curve (b) is for a non-head-on approach.

results based on a careful observation of Fig. 2 agree with the formula for the Doppler effect for a moving source:

$$f = f_0 \left(1 \pm v_s/v\right)^{-1}, \qquad (4)$$

where f is the frequency of the waves according to the observer, f_0 is the A more difficult case to analyze arises when the source, still moving at frequency the observer would detect if the source were at rest, and v_s/v is the ratio of the speed of the source to the speed of the waves. The minus sign is for an approaching source, and the plus sign for a receding one. In the present numerical example, we have assumed $v_s/v = 1/2$.

3d. Oblique Approach. A more difficult case to analyze arises when the source, still moving at constant velocity, does not approach the observer head-on. In this case, the frequency undergoes a more gradual change as the source passes the observer (see curve (b) in Fig. 3). The detailed shape of this curve can be determined from the wave pattern in Fig. 2. If we draw a horizontal line some distance above or below the horizontal through the center of the source we can measure the wave front spacing at a series of points along this line, and from this the observed frequency at each point. Here we assume that the observer travels along the horizontal line through the wave pattern. This is equivalent to our original assumption of a stationary observer and a moving source, in which case the wave pattern depicted would move horizontally past the observer.
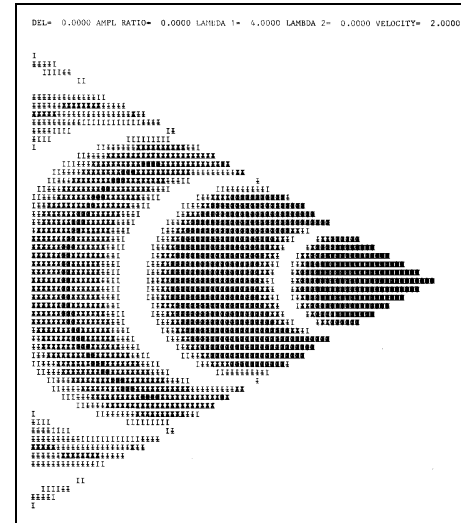
Figure 4. Computer generated intensity pattern for a shock wave produced when the source speed exceeds the wave speed.

3e. Shock Waves. A "shock wave" is produced when the source of waves travels at a speed v_s, which exceeds the speed of the waves v. As shown in the computer-generated snapshot in Fig. 4, the wave amplitude is zero outside a certain angular region (a cone in the three-dimensional case). This effect could manifest itself as the wake of a boat in the case of water waves, a sonic boom in the case of sound waves, or "Cerenkov radiation" in the case of light waves. The shock wave arises due to the superposition of waves emitted from the moving source at different times. Figure 5 shows a series of wave fronts emitted at five instants of time when the source is located at points A, B, C, D, and E. Constructive addition of the five wave fronts (as well as all others) occurs along a direction θ shown as the angle AFO. The line OF forms one edge of the shock wave. We see that the ratio $AO/AF = v/v_s$, from which we obtain

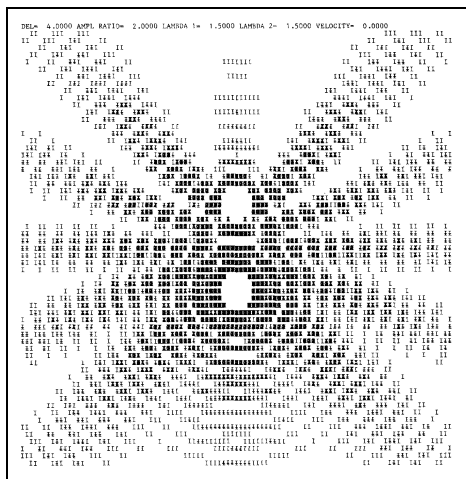$$\sin \theta = v/v_s, \qquad (5)$$

with v = wave speed, and v_s = source speed.

4. Superposition of Waves

4a. Coherent Addition of Waves. Sources are coherent if their relative phase does not vary in time. Two coherent point sources of waves will produce an "interference" pattern, such as the one shown in Fig. 6, as a consequence of the superposition principle according to which the

Figure 5. Wave fronts emitted from a moving source combining to produce a shock wave.

resultant wave amplitude at any point in the xy-plane is equal to the sum of the individual wave amplitudes, $A_1 + A_2$, at that point. An interference pattern arises, since the relative phase between A_1 and A_2 varies from point to point, due to the changing distances to each of the sources. Given two coherent point sources, separated by a distance d, which emit waves that are initially in phase, we may write for the amplitudes at a

Figure 6. Interference pattern for two point sources.

point (x, y), at time $t = 0$:

$$A_1(x, y) = \frac{S_1}{r_1} \sin\left[\frac{2\pi}{\lambda_1} r_1\right]$$

$$A_2(x, y) = \frac{S_2}{r_2} \sin\left[\frac{2\pi}{\lambda_2} r_2\right] \qquad (6)$$

where S_1 and S_2 are the source strengths, λ_1 and λ_2 are the wavelengths, and r_1 and r_2 are the distances from each source to the point (x, y). The intensity of a wave is proportional to the square of its amplitude. Therefore, at any point (x, y) we can find the resultant intensity from $I(x, y) = (A_1 + A_2)^2$. The intensity pattern shown in Fig. 6 was produced by explicitly finding $I(x, y)$ at a large number of points in a rectangular grid. For the interference pattern in Fig. 6, the sources are assumed to have equal strengths and equal wavelengths. In the usual treatment of this problem, the condition for destructive interference is that the waves from one source travel a distance which is an odd integral number of half-wavelengths more than the waves from the other source, i.e., $r_1 - r_2 = n\lambda/2$, $n = 1, 3, 5, \ldots$. However, this unrealistically ignores the inverse-square dependence of intensity on distance for each source. The computer-generated interference patterns can be made, as indicated in the preceding section, without using any such simplifying assumption. It would be expected, however, that at distances from each source that are large compared to the separation between sources d, there would be little difference between the r-dependent and r-independent treatment. In particular, we would expect that at distances from the sources that are large compared to their separation, the directions θ for which maxima and minima in the intensity are found, are given by the conditions:

$$\text{maxima: } n\lambda = d\sin\theta, \qquad \text{minima: } \left(n + \frac{1}{2}\lambda\right) = d\sin\theta \qquad (7)$$

where n is any integer. (The angle θ is defined with respect to the x-axis). Figure 6, for example, was generated using $d = 4$ and $\lambda = 1.5$. Equation (7) would therefore predict that maxima would be found along $\theta = 0°$, 22°, and 50°, which seems to be born out by inspection of the figure.

4b. Two Unequal Strength Sources. The usual analysis of interference restricts itself to sources of equal wavelengths and source strengths. There is no need to impose that simplifying restriction when we calculate intensity patterns directly in terms of two independent amplitudes. Interesting features appear when we examine patterns for unequal sources.
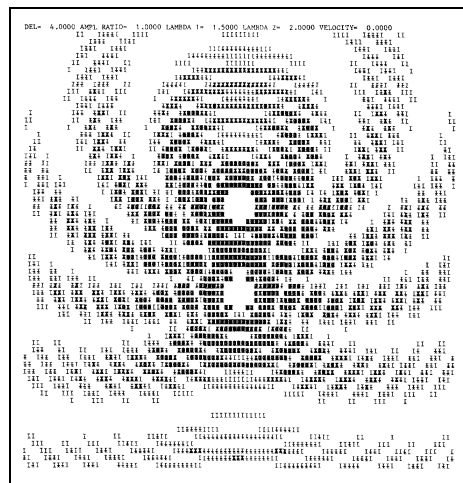
Figure 7. Interference pattern for two unequal strength point sources.

For example in the case of two sources of unequal strength (see Fig. 7) we find the interesting result that the interference maxima and minima are more pronounced on the side of the weaker source. The reason is that due to the inverse-square dependence of wave amplitude, A_1 and A_2 are more nearly equal in magnitude on the side of the weaker source. This permits complete destructive interference when the waves are out of phase at a given point, thereby yielding $I = (A_1 + A_2)^2 = 0$. The directions along which minima and maxima in the intensity occur at large distances are the same as for the case of equal source strengths. Thus, the points of maximum and minimum intensity lie on straight lines, except near the sources.

4c. Two Unequal Wavelength Sources. The intensity pattern from two coherent sources with different wavelengths λ_1 and λ_2 is shown in Fig. 8. Notice that the locus of points for minimum and maximum intensity are no longer straight lines. This feature of the pattern can be connected with the phenomenon known as "beats" which arises in the case of two sources having unequal frequencies or wavelengths. The combination of two frequencies f_1 and f_2 is found to give rise to a "beat frequency" $f = |f_1 - f_2|$. Remember that the wave patterns we have been examining are merely snapshots at some specific instant of time. To observe beats we need to consider how the waves from two sources add at a given point as a function of time. First consider the waves from two equal sources (Fig. 6). If we were to generate snapshots at successive instants of time we would find that the waves in that pattern seem to emanate outward

Figure 8. Interference pattern for two unequal wavelength sources

from the origin. At a given point on the x-axis, for example, successive crests and troughs pass that point as time progresses, but the maxima are always the same size wave after wave. Now consider how the situation might be different for two sources having unequal wavelengths or frequencies (Fig. 8). In this case the waves also seem to emanate from the origin. However, due to the curvature of the pattern we see that as we move along a specific line from the origin (say the x-axis) we go through successive regions of minimum and maximum intensity. This means that if we observe the intensity of the waves at a fixed point as a function of time we would find successive minima and maxima reaching this point as time progresses. That effect is precisely what is meant by beats.

4d. Two-Dimensional Fourier Analysis and Synthesis. Until now we have been primarily concerned with waves from point sources. It is also of interest to consider the interference of plane waves. For example, we can write for the amplitudes of two plane waves travelling along directions θ_1 and θ_2:

$$A_1(x, y) = S \sin \left[\frac{2\pi}{\lambda} (x \cos \theta_1 + y \sin \theta_1 - vt) \right]$$

$$A_2(x, y) = S \sin \left[\frac{2\pi}{\lambda} (x \cos \theta_2 + y \sin \theta_2 - vt) \right] \qquad (8)$$

For simplicity it has been assumed here that the two sources have equal strength S and wavelength λ. A time-frozen snapshot at t = 0 can be

obtained from the superposition principle A $= A_1 + A_2$, remembering that $I = (A_1 + A_2)^2$. It is also possible to superimpose a large number of plane waves having different wavelengths and source strengths. This is the basis of the two-dimensional "Fourier synthesis" of arbitrary intensity patterns. For example, any picture can be represented as a mathematical function $I(x, y)$ specifying the intensity at a large collection of points inside some region. The function $I(x, y)$ can then be synthesized using the appropriate combination of many plane waves, each weighted by the appropriate "Fourier coefficient." These Fourier coefficients need to be calculated for the specific two-dimensional function $I(x, y)$ that it is desired to synthesize. If a large enough number of waves are used (N), the Fourier synthesized function $I_N(x, y)$ and the original function $I(x, y)$ giving the point-by-point intensity of the picture will be indistinguishable. A primary reason for doing Fourier analysis and synthesis of pictures is "image enhancement," the improvement of picture quality by computer (an important goal in the space program and elsewhere). Once we have calculated the Fourier coefficients, for example, we can either enhance or suppress particular frequency ranges. This has the effect of either enhancing desired features in an image or suppressing unwanted ones, and it is quite analogous to adding treble or bass to give an altered frequency response in a hi-fi system.

5. Program to Produce Wave Patterns

5a. Input. The computer program that produced the wave patterns in this module requires five input parameters:

$$
\begin{aligned}
\texttt{DEL} &= \Delta(\text{separation between sources}) \\
\texttt{RATIO} &= S_2/S_1(\text{ratio of source strengths}) \\
\texttt{L1} &= \lambda_1(\text{wavelength for source 1}) \\
\texttt{L2} &= \lambda_2(\text{wavelength for source 2}) \\
\texttt{V} &= v_s/v(\text{ratio of the source and wave speeds}).
\end{aligned}
$$

To generate the single source patterns (Figs. 1, 2, and 4), `RATIO` must be set to zero, and to generate the interference patterns for two stationary sources (Figs. 6, 7, and 8), `V` must be set to zero. If there are two sources they are assumed to be located on the y-axis, symmetric about the origin. The scale used for all plots is that x and y are both assumed to be inside a square grid extending from -10 to $+10$. The input parameters used to generate the figures in this module are as follows:
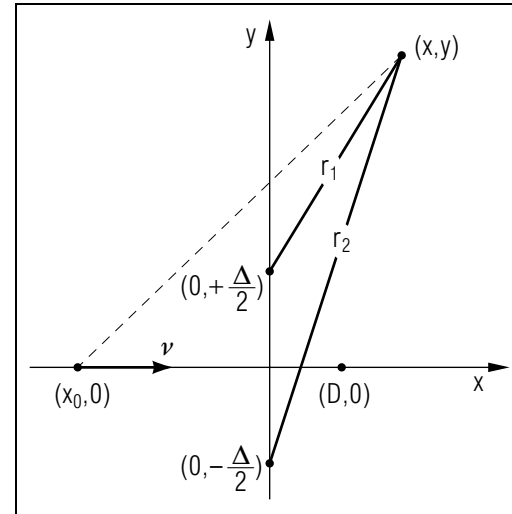
Figure 9. Locations of two point sources assumed in program.

	DEL	RATIO	L1	L2	V
Fig. 1	0.	0.	4.0	0.	0.
Fig. 2	0.	0.	4.0	0.	0.5
Fig. 4	0.	0.	4.0	0.	2.0
Fig. 6	4.0	1.0	1.5	1.5	0.
Fig. 7	4.0	2.0	1.5	1.5	0.
Fig. 8	4.0	1.0	1.5	2.0	0.

When running the BASIC version of the program you will also be asked to input values for two scaling variables, `XSCLE` and `YSCLE`, in a manner identical to that used in MISN-0-348.

5b. Calculating the Resultant Intensity. After the input parameters have been read, the program proceeds to calculate the resultant intensity (`RINT`) at all points within a rectangular grid of points inside the square region defined by $-10 \le x \le +10, -10 \le y \le +10$, in a manner similar to the method used in MISN-0-348 to calculate the electric potential on a grid. The points in the grid are identified by a pair of indices (j, k), where $j = 1, 2, \ldots$ indicates the row number, and $k = 1, 2, \ldots$ indicates the column number. At each grid point starting with $(j, k) = (1, 1)$, the program first computes the coordinates (x, y) in terms of the indices (j, k), using

$$x = \frac{1}{5}(k - 51)$$

$$y = \frac{1}{3}(j - 31). \tag{9}$$

If the two sources are at rest ($V = 0$), they are assumed to be symmetrically located, a distance Δ apart at the points $(0, -\Delta/2)$, $(0, +\Delta/2)$ (see Fig. 9). In this case, we can use the distance formula to obtain r_1 and r_2, the distances from each source to the point (x, y):

$$r_1 = \sqrt{x^2 + \left(y - \frac{\Delta}{2}\right)^2} \tag{10}$$

$$r_2 = \sqrt{x^2 + \left(y + \frac{\Delta}{2}\right)^2} \tag{11}$$

If source 1 is in motion the procedure for finding r_1 is more complicated. We assume that the source moves along the x-axis at constant speed and that at time $t = 0$ (the instant at which a wave front reaches the point (x, y)), the source is located at a particular point $(D, 0)$. We need to know the location of the source at the time when the wave front reaching (x, y) at $t = 0$ was emitted from the source (see Fig. 9). The technique for finding the location of the source $(x_0, 0)$ at the emission time is described in the Appendix. Once the value of x_0 has been determined, r_1 can then be obtained using the distance formula

$$r_1 = ((x - x_0)^2 + y^2)^{1/2}. \tag{12}$$

There is, however, one important exception. It may happen that no solution for x_0 exists. This occurs when v_s exceeds v, and the point (x, y) is outside the shock wave cone. In this event, the amplitude of the wave at the point (x, y) is zero. In order to treat this case in the program, using the same formula for the amplitude (Eq. 6), the program sets $r_1 = \lambda_1$, which achieves the desired result, because then $\sin(2\pi r_1/\lambda_1) = 0$. Once the program obtains r_1 and r_2 using either Eqs. (10) and (11) (if both sources are at rest), or Eq. (12) (if source 1 is in motion), it adds 0.000001 to r_1, r_2, and λ_2 so that when it computes the wave amplitudes A_1 and A_2 according to Eq. (6), it will never divide by zero. It finally computes the resultant intensity $\texttt{RINT} = A^2 = (A_1 + A_2)^2$. For reasons which will be clear later, the program sets $\texttt{RINT} = 0$ if the point (x, y) is very close to either source and $\texttt{V}$ is not zero. (This produces small blanked-out circular regions around each source, which serves to locate them.)

5c. Graphically Plotting the Intensities.   After finding the resultant intensity at one point, the program proceeds to the next point within

that row. Once all the points within a row have been considered, the array $\texttt{RINT(K)}$, $K = 1,2,...$, contains the resultant intensities. The next step is to digitize the intensity in each cell of the row and put these printable values into the array $\texttt{IMAGE}$.

In the BASIC and second C++ versions of the program, the program goes through the elements of each row, comparing each element of the array $\texttt{RINT}$ to the array of seven threshold intensities, $\texttt{THRES(L)}$, to determine the maximum intensity the element is at or above. It then adds a digit (0-6) to the array $\texttt{IMAGE}$, with higher digits indicating higher threshold intensities. Upon completion of one pass on the row, with each element of the array $\texttt{IMAGE}$ now having its threshold digit, the row is complete. The entire set of rows is sent to a disk file that can be sent directly to a printer or can be loaded into a word processing program and sent to a printer from there.

In the FORTRAN and first C++ versions, the program makes thirteen consecutive passes through each row of elements. On each of the seven odd numbered passes on a row, there is a progressively higher value for the threshold, $\texttt{THRES(L)}$. If the $\texttt{RINT}$ value for an element is at or above the threshold of the pass, a graphic character assigned to that pass is added to the array $\texttt{IMAGE}$. If the intensity is below the threshold for the pass, a blank character is added. On the even numbered passes, a backspace character is added for each element of the row. Thus the array of characters representing the row winds up having thirteen times as many elements as will be in the actual printed row. When printed, the six strings of backspaces between the the seven strings of characters causes the strings of characters to be printed directly on top of each other (overprinted). Thus the highest intensity points will be represented by seven different non-blank characters printed on top of one another and lower intensity points will be represented by fewer non-blank overprint characters (possibly none). At each position in the pattern, then, the number of overprinted characters, and therefore the darkness of the pattern, represents the intensity at that point. The entire set of rows is sent to a disk file that needs to be sent directly to a printer. If the file is loaded into a word processing program and sent to a printer from there, the word processing program will probably send the backspaces as forward spaces and that can produce a huge number of worthless printed pages.

6. A PROJECT TO PRODUCE WAVE PATTERNS

6a. One Moving Wave Source.

[1.] Enter and run the program to generate wave patterns corresponding to one source moving at various speeds. For example, the following five sets of data will generate patterns for a source with wavelength $\lambda = 4.0$ and a speed of 0.25, 0.50, 0.75, 1.00 and 2.00 times the wave speed:

DEL	RATIO	L1	L2	V
0.	0.	4.0	0.	0.25
0.	0.	4.0	0.	0.50
0.	0.	4.0	0.	0.75
0.	0.	4.0	0.	1.00
0.	0.	4.0	0.	2.00

[2.] For the first four cases verify the Doppler effect formula $f = f_o/(1 \pm v_s/v)$ by measuring the average distance between waves in front of the source and behind the source.

[3.] The average distance between waves is inversely proportional to the frequency f and according to the Doppler effect formula the ratio, R, of the distance between waves in front and back of the source should be given by:

$$R = \frac{1 - v_s/v}{1 + v_s/v}$$

See how well your measurements agree with this result for the first four cases.

[4.] The fifth set of data will produce a shock wave since the source speed exceeds the wave speed $v_s/v > 1$. See how well your results satisfy the condition for the half angle of the shock wave cone: $\sin\theta = v/v_s$. Note that on some printers the x and y dimensions may not be printed out according to the same scale. This will effect the measurement of angles: see Appendix A.

6b. Interference Between Two Point Sources.

[5.] Enter and run the program to generate wave patterns corresponding to two point sources of equal source strengths and wavelengths. Use

$\lambda_1 = 2$, and try the following four Δ values: $\Delta = 1.0, 2.0, 4.0, 8.0$. At distances from the sources that are large compared to their separation Δ the location of maxima and minima should be along straight lines given by Eq. (7). Make measurements on your output in each case, and see how well Eq. (7) is satisfied. Remember that angles measured on the output may be distorted if your printer does not give square plots. In that case, the actual angle θ can be found from the measured angle θ' using Eq. (14).

[6.] The largest Δ value ($\Delta = 8$) should give a plot in which the character of the minima and maxima closer to the sources can be more easily examined. For this case you should find that the locus of maxima or minima consist of hyperbolae in the vicinity of the sources. Draw these hyperbolae directly on the output, showing the expected places where maxima and minima should be found.

6c. Intrinsic Phase Shifts.

[7.] One simple modification you can make to the program is to introduce an intrinsic phase difference ϕ to one of two interfering sources. This would have the effect of altering the two amplitudes given in Eq. (6) to yield:

$$A_1(x,y) = \frac{S}{r_1} \sin\left(\frac{2\pi}{\lambda_1} r_1\right),$$

$$A_2(x,y) = \frac{S}{r_2} \sin\left(\frac{2\pi}{\lambda_2} r_2 + \phi\right).$$

Make this modification to the program, and try running one of the cases in Sect. 6b using $\phi = \pi$ radians (or 180°) and $\phi = \pi/2$ radians (or 90°). Describe the effect that the introduction of this intrinsic phase shift has on the resulting wave patterns.

6d. Interference of Incoherent Sources.

[8] Another simple modification you can make to the program is to make the two sources add incoherently rather than coherently. Whether the sources are coherent or incoherent the resultant intensity is given by:

$$I = (A_1 + A_2)^2 = A_1^2 + A_2^2 + 2 A_1 A_2. \tag{13}$$

What happens with an incoherent source is that the relative intrinsic phase difference ϕ between the sources constantly changes in a random

manner. The sign of the term $A_1 A_2$ in Eq. (13) is therefore subject to continual and rapid fluctuations. On a long-term basis it therefore averages to zero, and for the resultant intensity for two incoherent sources we have:

$$I = A_1^2 + A_2^2.$$

Modify the program to treat the case of two incoherent sources and run it using one of the data sets you used in Sect. 6b. You should find that the interference maxima and minima no longer exist.

Acknowledgments

A. Correcting Angles on Plots

Suppose that for the first four graphs the ratio of the width (x) to the height (y) of the plotted pictures is not exactly 1.0, but is instead a factor r, (called the "aspect ratio"). In that case a line which makes an angle θ with the x-axis will be printed out making a different angle θ'. The relationship between the actual and measured angles is:

$$\theta = \arctan(r \tan \theta'). \tag{14}$$

Be sure to make this angle correction if you find that your printer does not produce square plots.

B. Source Position with No Wave Front

We wish to determine the position $(x_0, 0)$ of a moving source at the time when the wave front reaching an observer at (x, y) and $t = 0$ was being emitted from the source ($t = -T$). As can be seen in Fig. 9, the distance traveled by the wave to reach the point (x, y) from the point $(x_0, 0)$ is

$$r = v\, T = ((x - x_0)^2 + y)^{1/2},$$

where v is the wave speed. If the source is located at a known point $(D, 0)$ at $t = 0$, then we may write

$$D - x_0 = v_s\, T,$$

where v_s is the source speed. Eliminating T between the above pair of equations, we obtain

$$\frac{v}{v_s}(D - x_0) = ((x - x_0)^2 + y^2)^{1/2})$$

With the aid of a little algebra, the above equation can be written as a quadratic equation in x_0:

$$A\, x_0^2 + B\, x_0 + C = 0$$

where:

$$A = 1 - (\frac{v}{v_s})^2$$

$$B = \frac{2Dv^2}{v_s^2} - 2x$$

$$C = x^2 + y^2 - (\frac{Dv}{v})^2$$

Note that no solution x_0 exists if $B^2 - 4AC < 0$; this occurs when $v_s/v > 1$ and the point (x, y) lies outside the Mach cone. Otherwise, we can solve for x_0 using the quadratic formula:

$$x_0 = \frac{-B + (B^2 - 4AC)^{1/2}}{2A}$$

We reject the other root, since it corresponds to a point which lies to the right of $(D, 0)$ (see Fig. 9).

C. Fortran, Basic, C++ Programs

All programs are at

http://www.physnet.org/home/modules/support_programs

which can be navigated to from the home page at

http://www.physnet.org

by following the links: → modules → support programs, where the programs are:

m353p1f.for, Fortran; (uses overprinting to indicate intensity)
m353p1b.bas, Basic; (uses numbers to indicate intensity)
m353p1c.cpp, C++; (uses overprinting to indicate intensity)
m353p2c.cpp, C++; (uses numbers to indicate intensity)
lib351.h, needed Library for C++ program;

MODEL EXAM

1. See Output Skills K1

Examinee:
On your computer output sheet(s):

  (i) Mark page numbers in the upper right corners of all sheets.

 (ii) Label all output, including all axes on all graphs.

On your Exam Answer Sheet(s), for each of the following parts of items (below this box), show:

  (i) a reference to your annotated output; and

 (ii) a blank area for grader comments.

When finished, staple together your sheets as usual, but include the original of your annotated output sheets just behind the Exam Answer Sheet.

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2. Submit your hand-annotated output showing wave intensity plots for a point source moving at various speeds showing:

   a. your measurements (on your output) of the average distance between waves in front of and behind the source and the extent to which they agree with the Doppler formula, for four non-shock-wave cases.

   b. your measurements (on your output) and the extent to which they agree with the theoretical value for the half-ange of the shock wave cone for a case where the source speed exceeds the wave speed.

3. Submit your hand-annotated output showing wave intensity plots for two interfering point sources for at least four values of the inter-source separation, showing:

   a. your measurements (on your output) of the positions of the maxima and minima for these cases and the extent to which they agree with the theoretical values.

   b. the extent to which the locii of the maxima and minima near the sources (for large inter-source separation) are hyperbolae.

4. Submit your hand-annotated output showing wave intensity plots for two interfering point sources showing the effect of a shift in the phase of one of the sources, showing:

   a. a comparison of the un-phase-shifted case and the cases of 90° and 180° phase shifts.

   b. the qualitative reason for the differences one sees.

5. Submit your hand-annotated output output showing wave intensity plots for two interfering incoherent point sources, showing:

   a. a comparison of a coherent case and the same case but with the intrinsic relative phase between the two sources constantly changing in a random manner.

   b. the qualitative reason for the differences one sees.

INSTRUCTIONS TO GRADER

If the student has submitted copies rather than originals of the computer output, state that on the exam answer sheet and **immediately stop grading the exam and give it a grade of zero.**