PHYSNET

MISN-0-34

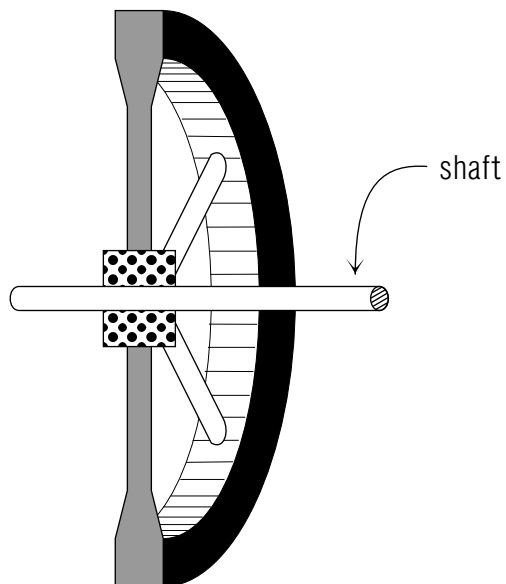# TORQUE AND ANGULAR MOMENTUM IN CIRCULAR MOTION

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TORQUE AND ANGULAR MOMENTUM IN CIRCULAR MOTION

by

Kirby Morgan, Charlotte, Michigan

Title: **Torque and Angular Momentum in Circular Motion**

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Input Skills:

1. Vocabulary: kinetic energy (MISN-0-20), torque, angular acceler-ation (MISN-0-33), angular momentum (MISN-0-41).
2. Solve constant angular acceleration problems involving torque, moment of inertia, angular velocity, rotational displacement and time (MISN-0-33).
3. Justify the use of conservation of angular momentum to solve prob-lems involving torqueless change from one state of uniform circular motion to another (MISN-0-41).

Output Skills (Knowledge):

K1. Define the torque and angular momentum vectors for (a) a single particle (b) a system of particles.

K2. Starting from Newton's 2nd law, derive its rotational analog and state when it can be written as a scalar equation.

K3. Start from the equation for linear kinetic energy and derive the corresponding one for rotational kinetic energy.

K4. Explain why conservation of angular momentum may not hold in one system but may if the system is expanded.

Output Skills (Problem Solving):

S1. For masses in circular motion at fixed radii, solve problems relating torque, moment of inertia, angular acceleration, rotational kinetic energy, work, and angular momentum.

S2. Given a system in which angular momentum is changing with time due to a specified applied torque, reconstruct the minimum ex-panded system in which total angular momentum is conserved. Describe the reaction torque which produces the compensating change in angular momentum.

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TORQUE AND ANGULAR MOMENTUM IN CIRCULAR MOTION

by

Kirby Morgan, Charlotte, Michigan

1. Introduction

Just as for translational motion (motion in a straight line), circular or rotational motion can be separated into kinematics and dynamics. Since rotational kinematics is covered elsewhere,[1] the discussion here will center on rotational dynamics. Our goal is to derive the rotational analog of Newton's second law and then apply it to the circular motion of a single particle and to systems of particles. In particular we wish to develop the relationship between torque and angular momentum and discuss the circumstances under which angular momentum is conserved.

2. Torque and Angular Momentum

2a. Definitions. The torque and angular momentum are defined as vector products of position, force and momentum. Suppose a force $\vec{F}$ acts on a particle whose position with respect to the origin O is the displacement vector $\vec{r}$. Then the torque "about the point 0 and acting on the particle," is defined as:[2]

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{1}$$

Now suppose the particle has a linear momentum P relative to the origin. Then the angular momentum of the particle is defined as:

$$\vec{L} = \vec{r} \times \vec{p}. \tag{2}$$

The directions of $\vec{\tau}$ and $\vec{L}$ are given by the right-hand rule for cross products (see Fig. 1).

2b. Relationship: $\vec{\tau} = d\vec{L}/dt$. Using the definitions of torque and angular momentum, we can derive a useful relationship between them.

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[1] See "Kinematics: Circular Motion" (MISN-0-9) and "Torque and Angular Acceleration for Rigid Planar Objects: Flywheels" (MISN-0-33).
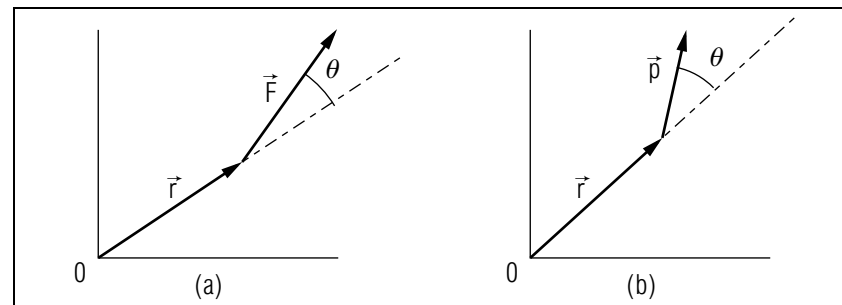
[2] See "Force and Torque" (MISN-0-5).

Figure 1. Vector relationships for: (a) torque (b) angular momentum (both directed out of the page).

Starting from Newton's second law, written in the form

$$\vec{F} = \frac{d\vec{p}}{dt}, \tag{3}$$

the torque is:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}. \tag{4}$$

This can be rewritten using the expression for the derivative of a cross product: *Help: [S-1]*

$$\vec{\tau} = \frac{d}{dt}(\vec{r} \times \vec{p}) - \frac{d\vec{r}}{dt} \times \vec{p}$$
$$= \frac{d\vec{L}}{dt} - \vec{v} \times \vec{p}. \tag{5}$$

Now $\vec{p} = m\vec{v}$ so $\vec{v} \times \vec{p} = 0$ (because the vector product of parallel vectors is zero), so the torque is:[3]

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \tag{6}$$

Thus the time rate of change of the angular momentum of a particle is equal to the torque acting on it.

2c. Motion Confined to a Plane. The expression $\vec{\tau} = d\vec{L}/dt$ for a particle takes on a scalar appearance when the motion of the particle is confined to a plane.

Consider a particle constrained to move only in the x-y plane, as shown in Fig. 2. The torque on the particle is always perpendicular to

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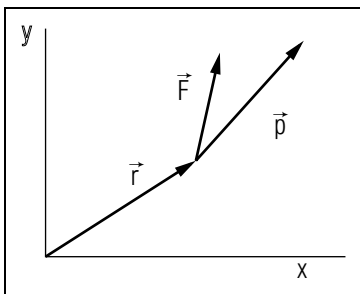[3] This equation is valid only if $\vec{\tau}$ and $\vec{L}$ are measured with respect to the same origin.

Figure 2. $\vec{r}$, $\vec{F}$, and $\vec{p}$ are all coplanar for motion in a plane.
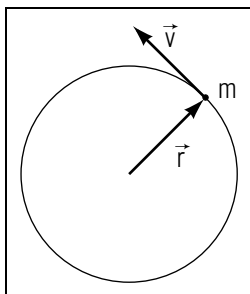

Figure 3. For a particle in circular motion, $\vec{r}$ and $\vec{v}$ are perpendicular.

this plane as is the angular momentum [work this out using Eqs. (1) and (2)]. Equivalently, we say that $\vec{\tau}$ and $\vec{L}$ have only z-components. Since their directions remain constant, only their magnitudes change. Then:

$$\tau = \frac{dL}{dt} \qquad \text{(motion in a plane)}. \qquad (7)$$

This equation holds only if $\vec{F}$ and $\vec{p}$ are in the same plane; if not (and they won't be for non-planar motion), the full Eq. (6) must be used.[4]

2d. Circular Motion of a Mass. The torque and angular momentum for the special case of a single particle in circular motion can be easily related to the particle's angular variables. Suppose a particle of mass m moves about a circle of radius r with speed v (not necessarily constant) as shown in Fig. 3. The particle's angular momentum is:

$$\vec{L} = \vec{r} \times m\vec{v}, \qquad (8)$$

but since $\vec{r}$ and $\vec{v}$ are perpendicular,[5] the magnitude of $\vec{L}$ is:

$$L = mvr \qquad (9)$$

and the direction is out of the page. Equation (9) may be rewritten in terms of the angular velocity (since $v = \omega r$) as:

$$L = mr^2\omega. \qquad (10)$$

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[4]The component equations are $\tau_x = dL_x/dt$, $\tau_y = dL_y/dt$, $\tau_z = dL_z/dt$.

[5]See "Kinematics: Circular Motion" (MISN-0-9).

Similarly, the torque is:

$$\tau = \frac{dL}{dt} = mr^2 \frac{d\omega}{dt} = mr^2\alpha, \qquad (11)$$

where α is the particle's angular acceleration.[6]

3. Systems of Particles

3a. Total Angular Momentum. The total angular momentum of a system of particles is simply the sum of the angular momenta of the individual particles, added vectorially. Let $\vec{L}_1$, $\vec{L}_2$, $\vec{L}_3$, ..., $\vec{L}_N$, be the respective angular momenta, about a given point, of the particles in the system. The total angular momentum about the point is:[7]

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots = \sum_{i=1}^{N} \vec{L}_i. \qquad (12)$$

As time passes, the total angular momentum may change. Its rate of change, $d\vec{L}/dt$, will be the sum of the rates $d\vec{L}_i/dt$ for the particles in the system. Thus $d\vec{L}/dt$ will equal the sum of the torques acting on the particles.

3b. Total Torque. The total torque on a system of particles is just the sum of the external torques acting on the system. The torque due to internal forces is zero because by Newton's third law the forces between any two particles are equal and opposite and directed along the line connecting them. The net torque due to each such action-reaction force pair is zero so the total internal torque must also be zero. Then the total torque on the system is just equal to the sum of the external torques:

$$\vec{\tau} = \sum_{i=1}^{N} \vec{\tau}_{i,\text{ext}} \qquad \text{(system of particles)}. \qquad (13)$$

For the system, then:

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \qquad (14)$$

In words, the time rate of change of the total angular momentum about a given point, for a system of particles, is equal to the sum of the external torques about that point and acting on the system.

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[6]See "Torque and Angular Acceleration for Rigid Plane Objects: Flywheels" (MISN-0-33).

[7]For continuous mass distributions the summation becomes an integration.

3c. Rigid Body Motion About a Fixed Axis. A rigid body is a system of particles whose positions are all fixed relative to each other. Since $L = mvr = mr^2\omega$ for each particle in the body, we may write for the total angular momentum:

$$L = \left(\sum_i m_i r_i^2\right)\omega, \tag{15}$$

where we have assumed that the body is rotating about a fixed axis with angular velocity ω. The quantity in parentheses,

$$I \equiv \sum_i m_i r_i^2, \tag{16}$$

is called the "moment of inertia" of the body.[8] Thus we write

$$L = I\omega \tag{17}$$

and, since the axis of rotation is fixed, equations with a scalar appearance hold for the torque:

$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha. \tag{18}$$

3d. Example: Flywheel. As a visual example, we calculate the torque and angular momentum for the special case of a flywheel where the entire mass is uniformly distributed around the rim. Let "dm" be the mass of an infinitesimal segment of the rim as shown in Fig. 4. The angular momentum dL of the mass dm is:

$$dL = dm\, r^2\omega. \tag{19}$$

Since r and ω are the same for all points on the rim, the total angular momentum for the flywheel is:

$$L = \int dL = r^2\omega \int dm = Mr^2\omega = I\omega, \tag{20}$$

where M is the total mass of the flywheel. This is identical to Eq. (17) for a single mass M in circular motion. The total torque is also the same, i.e.,

$$\tau = \frac{dL}{dt} = Mr^2\alpha = I\alpha. \tag{21}$$

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[8]See "Uniform Circular Motion: Moment of Inertia, Conservation of Angular Momentum, Kinetic Energy, Power" (MISN-0-41).
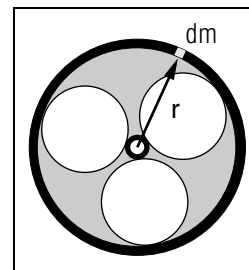
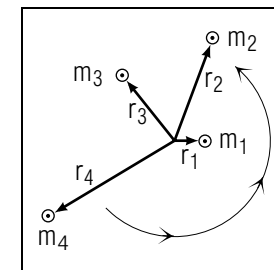**Figure 4.** Mass on the rim of a flywheel (heavy line).

Figure 5. Rotational E_k equals the sum of particles' E_k.

3e. Kinetic Energy of Rotation. The total kinetic energy of a system can be written in terms of the system's moment of inertia and angular velocity. We start with the statement that the total kinetic energy of the system of particles, each of which is in circular motion about a fixed axis of rotation, is equal to the sum of the kinetic energies of the individual particles.

Each individual particle of mass m_i moves in a circle of radius r_i about the axis of rotation. If the positions of the particles are all fixed relative to each other (as in a rigid body), then the angular velocity ω is the same for all particles. The kinetic energy of each particle is thus:

$$E_{ik} = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2\omega^2$$

The total kinetic energy of the rotating body is therefore

$$E_k = \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \ldots = \frac{1}{2}\left[\sum_{i=1}^N m_i r_i^2\right]\omega^2$$

By Eq. (16) the sum, $\sum m_i r_i^2$, is just the moment of inertia I of the body, so the rotational kinetic energy for a rigid body can finally be written:

$$E_k = \frac{1}{2}I\omega^2. \tag{22}$$

▷ See if you can determine the moment of inertia and kinetic energy of our flywheel in Sect. 3d. *Help: [S-2]*

3f. Linear vs. Rotational Motion. Here is a comparison of the equations of dynamics in "normal" and "rotational" form:

Table 1. General forms.	
Normal	Rotational
$\vec{F} = d\vec{P}/dt$	$\vec{\tau} = d\vec{L}/dt$
$E_k = MV^2/2$	$E_k = I\omega^2/2$

For a rigid body rotating about a fixed axis, the following comparisons hold.

Table 2. Rigid-body forms.	
Normal	Rotational
$P = Mv$	$L = I\omega$
$F = dp/dt = Ma$	$\tau = dL/dt = I\alpha$

Some of the equations shown in the Table 2 can be rewritten in vector/tensor form so they have validity beyond linear and circular motion.

4. Conservation of Angular Momentum

4a. Statement of the Law. The total angular momentum of a system of particles is conserved if there are no external torques acting on the system. Thus:

$$\vec{\tau} = 0 = \frac{d\vec{L}}{dt} \qquad (23)$$

so $\vec{L}$ is a constant. For a rigid body rotating about a fixed axis, $L = I\omega$, so that if I changes, there must be a compensating change in ω in order for L to remain constant.

4b. If the External Torque is not Zero. If the external torque on a system is not zero then angular momentum is not conserved for the system. However, all is not lost, for if the system is expanded to include whatever is causing the external torque on it (therefore changing it into an internal torque), angular momentum will be conserved for the expanded system.

4c. Example: Two Flywheels. Consider what happens when two identical flywheels, one spinning and one at rest, are suddenly brought together. The spinning flywheel has total angular momentum $L_0 = Mr^2\omega_0$ before it is allowed to come in contact with the second flywheel. When the two flywheels come together, the friction between them causes a torque
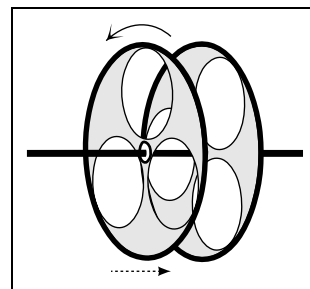
Figure 6. Two flywheels, one spinning, one at rest, are brought together.

to be exerted on the spinning one and, since this torque comes from an external source, the angular momentum of the spinning flywheel is not conserved. If, however, our system consists of both flywheels, the torques each exert on the other are internal to the system and so there is no longer an external torque. Thus for the expanded system, containing both flywheels, angular momentum is conserved. This tells us that:

$$Mr^2\omega_0 = Mr^2\omega_f + Mr^2\omega_f = 2Mr^2\omega_f, \qquad (24)$$

where ω_f is the final angular velocity at which the two flywheels rotate. Once they are together—the spinning one slowed down ($\omega_0 \to \omega_f$), the other having gone from zero to ω_f—the two act as a single flywheel, with mass $2M$, rotating with angular velocity $\omega_f = \omega_0/2$. *Help: [S-3]*.

4d. Kinetic Energy of the Two Flywheels. When flywheels rotating at different speeds are brought into contact, it can be expected that kinetic energy is not conserved. This is because the nonconservative frictional force acts on the two flywheels. The kinetic energy is initially

$$E_{k0} = \frac{1}{2}Mr^2\omega_0^2, \qquad (25)$$

and afterward it is: *Help: [S-4]*

$$E_{kf} = Mr^2\omega_f^2 = \frac{1}{4}Mr^2\omega_0^2. \qquad (26)$$

Therefore the ratio of the final kinetic energy to the initial kinetic energy is:

$$\frac{E_{kf}}{E_{k0}} = \frac{1}{2}. \qquad (27)$$

This means that half of the initial kinetic energy has been dissipated due to the frictional forces between the surfaces of the two flywheels. Thus the total angular momentum of the system is conserved even though the total kinetic energy is not.

5. Nonplanar Rigid Bodies

Although the flywheel has been used as a specific example of rotational motion, the concepts described can be applied to all rigid bodies whether they are planar or not. The rotational motion of nonplanar rigid bodies is discussed elsewhere.[9]

Acknowledgments

Glossary

- **angular momentum (vector)**: $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{p}$ is the linear momentum of a particle at a position $\vec{r}$ with respect to the origin.

- **conservation of angular momentum**: the total angular momentum of a system is conserved if the external torque on the system is zero.

- **moment of inertia (of a system of particles)**: the sum $I \equiv \sum m_i r_i^2$.

- **rigid body**: a system of particles whose positions are fixed relative to each other.

- **rotational dynamics**: the application of dynamics to rotating bodies.

- **rotational kinetic energy**: the kinetic energy of a particle or system of particles rotating about a fixed point.

- **torque (vector)**: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{F}$ is the force acting on a particle located at $\vec{r}$.

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[9]See "Rotational Motion of a Rigid Body" (MISN-0-36).

PROBLEM SUPPLEMENT

Problem 5 also occurs on this module's *Model Exam*.

1. A flywheel of radius 0.50 m and mass 500.0 kg has a torque of 30.0 N m acting on it.

   a. Calculate its moment of inertia.

   b. Find its angular acceleration.

   c. Assuming the flywheel is initially at rest, calculate the time required to complete the first two revolutions.

   d. What is its angular velocity and tangential velocity after two revolutions?

   e. Find its kinetic energy after two revolutions.

2. A very lightweight circular platform has a weight of 300.0 N placed on it at a distance 25 cm from its center. The platform is placed horizontally on a pedestal such that a frictional drag force acts on the platform at the point where the weight is. The coefficient of friction is $\mu = 0.050$.

   a. Sketch the forces acting on the platform (assume it has zero mass) and find their numerical values.

   b. Calculate and sketch all torques acting on the system.

   c. If the platform initially rotates at 8π radians/s, find how long it takes for it to slow down and stop.

   d. What else must be included in the system in order for angular momentum to be conserved?

3. Suppose the stabilizing gyroscope of a ship has a rotor of mass 5.0×10^4 kg, all located on the rim at a radius of 0.20 m. The rotor is started from rest by a constant force of 1.00×10^3 N applied through a belt on the rim by a motor.

   a. Draw a diagram showing all forces acting on the rotor.

   b. Show all torques acting on the rotor.

   c. Compute the length of time needed to bring the rotor up to its normal speed of 9.00×10^2 rev/min.

d. State what else you must include in the system so that the total angular momentum will be conserved. That is, name the other object whose rotational speed about the rotor shaft changes oppositely as the rotor picks up speed. Describe the reaction torque which produces the angular acceleration of the other object.

4. A system consists of two massless struts, rigidly connected to a sleeve as shown in the diagram. The force $\vec{F}$ is always at right angles to its strut and the axle is vertical, enabling the system to rotate freely in a horizontal plane (the sketch is a view "looking down"). Neglect gravity.
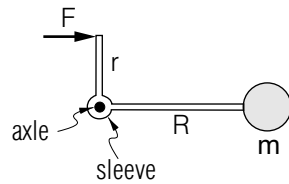
a. State why gravity can be neglected in this problem.

b. Sketch all forces acting on the system.

c. Write down and justify the value of the (net) total force acting on the system.

d. Write down or sketch all torques acting on the system.

At the end of the system's first complete revolution, derive the mass m's:

e. moment of inertia about the axle

f. angular acceleration

g. time

h. angular velocity

i. tangential velocity

j. kinetic energy

k. work done on it ($\int \vec{F} \cdot d\vec{x}$)

l. total energy

m. change in angular momentum about the axle.

Describe a plausible expanded system within which angular momentum is conserved in the above case. Specifically:

n. Describe the reaction torque which produces the compensating angular momentum.

5. A vertical flywheel of radius R contains (virtually) all of its mass M on its rim and has a handle on one spoke at a radius r which is less than the radius of the rim. You stand next to the flywheel, grasp the handle, and apply a constant tangential force F to the handle.

a. Sketch all forces acting on the flywheel including that due to gravity.

b. Write down and justify the value of the net (total) force on the flywheel.

c. Write down or sketch all torques acting on the system.

At the end of one-half of a revolution, find the rim's:

d. moment of inertia about the axle

e. angular acceleration

f. time

g. angular velocity

h. tangential velocity

i. kinetic energy

j. work done on it

k. total energy

l. angular momentum

m. Describe the mechanism by which angular momentum is conserved in this case.

Brief Answers:

1. a. $I = \sum m_i r_i^2 = Mr^2 = (500\,\text{kg})(0.50\,\text{m})^2 = 125\,\text{kg}\,\text{m}^2$

b. $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{30\,\text{N}\,\text{m}}{125\,\text{kg}\,\text{m}^2} = 0.24\,\text{rad/s}^2$$

c. $\theta = \frac{1}{2}\alpha t^2$: $\theta = 4\pi$

$$t = \left(\frac{2\theta}{\alpha}\right)^{1/2} = \left[\frac{(2)(4\pi)}{0.24\,\text{s}}\right]^{1/2} = 10.2\,\text{s}$$

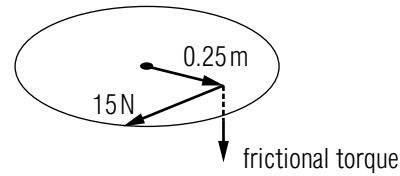d. $\omega = \omega_0 + \alpha t = 0 + (0.24/\text{s}^2)(10.2\,\text{s}) = 2.4\,\text{rad/s}$

$v = \omega r = (2.4\,\mathrm{rad/s})(0.50\,\mathrm{m}) = 1.2\,\mathrm{m/s}$

e. $E_k = \dfrac{1}{2}I\omega^2 = \dfrac{1}{2}(125\,\mathrm{kg\,m^2})(2.4\,\mathrm{s})^2 = 360\,\mathrm{J}$

2. a. $F(\,\mathrm{gravity}) = 300\,\mathrm{N}$
   $F(\,\mathrm{reaction}) = 300\,\mathrm{N}$

   $F(\,\mathrm{friction}) = \mu N(\mathrm{normal})$
   $\qquad\quad = (0.05)(300\,\mathrm{N})$
   $\qquad\quad = 15\,\mathrm{N}$
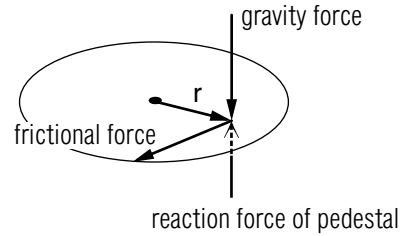
   b. $\tau(\,\mathrm{friction}) = rF(\,\mathrm{friction})$
   $\qquad\quad = (0.25\,\mathrm{m})(15\,\mathrm{N})$
   $\qquad\quad = 3.75\,\mathrm{N\,m}$

   c. $\tau = I\alpha;\ I = Mr^2$

   $\alpha = \dfrac{\tau}{I} = \dfrac{\tau}{Mr^2} = \dfrac{3.75\,\mathrm{N\,m}}{(300\,\mathrm{N}/9.8\,\mathrm{m/s^2})(0.25\,\mathrm{m})^2} = 1.96\,/\mathrm{s^2}$

   $\omega = \alpha t$

   $t = \dfrac{\omega}{\alpha} = \dfrac{8\pi/\mathrm{s}}{1.96/\mathrm{s^2}} = 12.8\,\mathrm{s}$

   d. If the pedestal and earth are included then the earth will acquire the compensating angular momentum so angular momentum is conserved for the combined system.

3. a.
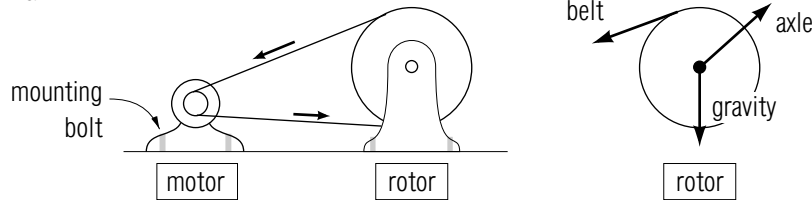
   b. only one torque (from belt): out of paper.

   c. $1.57 \times 10^1\,\mathrm{min}$.

   d. The motor exerts a torque on the ship through the motor's mounting bolts and mounting bed, while the ship transmits it to the water and the water transmits some of it to the earth.

4. a. The force and torque produced by gravity are exactly cancelled by the force and torque exerted on the system to keep it in a horizontal plane.

   b.
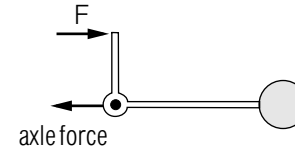
   c. net force is zero because the system goes nowhere (does not accelerate away from its present location).

   d. $\vec{\tau} = rF$, into the paper

   e. $I = mR^2$

   f. $\alpha = \tau/I = Fr/(mR^2)$

   g. $\theta = \theta_0 + \omega_0 t + \dfrac{1}{2}\alpha t^2$

   $2\pi = 0 + 0 + \left(\dfrac{1}{2}\right)\left(\dfrac{Fr}{mR^2}\right)t^2$

   $t = \left(\dfrac{4\pi m R^2}{Fr}\right)^{1/2}$

   h. $\omega = \omega_0 + \alpha t = 0 + \dfrac{Fr}{mR^2}\left(\dfrac{4\pi m R^2}{Fr}\right)^{1/2} = \left(\dfrac{Fr4\pi}{mR^2}\right)^{1/2}$

   i. $v = \omega R = \left(\dfrac{Fr4\pi}{m}\right)^{1/2}$
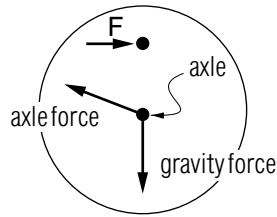
   j. $E_k = \dfrac{1}{2}mv^2 = Fr2\pi$

   k. $W = \int \vec{F}\cdot d\vec{x} = \int Fr\,d\theta = Fr2\pi$

   l. $E_{\mathrm{tot}} = E_k = Fr2\pi$

   m. $\Delta L = mvR = (mFr4\pi)^{1/2}R$ (into paper)

   n. If the axle is attached to the earth and if F's reaction force is against the earth, then it is the earth which acquires the compensating angular momentum about the axle.

5. a.

b. zero; no acceleration of the (center of the) flywheel.

c. Fr, into page

d. MR^2

e. $Fr/(MR^2)$

f. $\left(\dfrac{2\pi MR^2}{Fr}\right)^{1/2}$

g. $\left(\dfrac{Fr2\pi}{MR^2}\right)^{1/2}$

h. $\left(\dfrac{Fr2\pi}{M}\right)^{1/2}$

i. $Fr\pi$

j. $Fr\pi$

k. $Fr\pi$

l. $(MFr2\pi)^{1/2}\,R$

m. The expanded system consists of the flywheel, you, and the earth. The (frictional) force of your feet and the reaction force of the mounting bolts against the earth causes the earth's angular momentum about the axle to change in a compensating fashion.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-2b)*

$$\frac{d(\vec{r}\times\vec{p})}{dt} = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

so:

$$\vec{r}\times\frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) - \frac{d\vec{r}}{dt}\times\vec{p}$$

S-2 *(from TX-3e)*

$$I = r^2\int dm = Mr^2$$

$$E_k = \frac{1}{2}I\omega^2 = \frac{1}{2}Mr^2\omega^2$$

S-3 *(from TX-4c)*

$$Mr^2\omega_0 = Mr^2\omega_f + Mr^2\omega_f = 2Mr^2\omega_f$$

$$\implies \omega_0 = 2\omega_f$$

$$\omega_f = \frac{1}{2}\omega_0$$

S-4 *(from TX-4d)*

$$E_{kf} = \frac{1}{2}Mr^2\omega_f^2 + \frac{1}{2}Mr^2\omega_f^2 = Mr^2\omega_f^2 = Mr^2(\frac{1}{2}\omega_0)^2 = \frac{1}{4}Mr^2\omega_0^2$$

$$\implies \frac{E_{kf}}{E_{k0}} = \frac{\frac{1}{4}Mr^2\omega_0^2}{\frac{1}{2}Mr^2\omega_0^2} = \frac{1}{2}$$

MODEL EXAM

1. See Output Skills K1-K4 in this module's *ID Sheet*. The actual exam may contain one or more, or none, of these skills.

2. A vertical flywheel of radius R contains (virtually) all of its mass M on its rim and has a handle on one spoke at a radius r which is less than the radius of the rim. You stand next to the flywheel, grasp the handle, and apply a constant tangential force F to the handle.

    a. Sketch all forces acting on the flywheel including that due to gravity.

    b. Write down and justify the value of the net (total) force on the flywheel.

    c. Write down or sketch all torques acting on the system.

    At the end of one-half of a revolution, find the rim's:

    d. moment of inertia about the axle

    e. angular acceleration

    f. time

    g. angular velocity

    h. tangential velocity

    i. kinetic energy

    j. work done on it

    k. total energy

    l. angular momentum

    m. Describe the mechanism by which angular momentum is conserved in this case.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 5.