PHYSNET

MISN-0-29

DAMPED MECHANICAL OSCILLATIONS

x

spring (stiffness, k)

mass, m

shock absorber
(damping, λ)

frictionless surface

DAMPED MECHANICAL OSCILLATIONS

by

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Title: **Damped Mechanical Oscillations**

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Input Skills:

1. Work with Simple Harmonic Motion equations, especially the differential equation of motion (MISN-0-26).
2. Use and understand complex algebra and complex functions, especially $\exp(x + iy)$ type functions (MISN-0-59).

Output Skills (Knowledge):

K1. Vocabulary: underdamped oscillator, critically damped oscillator, overdamped oscillator, oscillation envelope.

K2. Write down the displacement equation solutions for underdamped, critically damped, and overdamped oscillators. Show that each is, in fact, a solution to Newton's Second Law for the force $F = -kx - \lambda v$, for the proper relationships of γ and ω to m, k, and λ.

K3. Sketch a graph of the displacement, $x(t)$ for the underdamped oscillator, labeling the envelope of the oscillations and indicating the initial phase used (not zero).

K4. Sketch graphs of displacement vs. time showing the transition from small to large damping, including the critically damped case.

Output Skills (Problem Solving):

S1. Given the decrease of an amplitude with time, find the damping constant and vice versa.

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1. Introduction

1a. Damping is Universal. In this module we add damping, or kinetic energy dissipation, to the case of simple harmonic motion. This is important because such dissipation is always present in real mechanical systems. In addition, we often wish to design the rate of dissipation in order to damp out unwanted oscillatory motions. An example of this is the up and down motion of a car wheel after it goes over a bump; without damping the wheel would continue to oscillate up and down indefinitely.

1b. An Example. In Fig. 1 we show a simple damped oscillatory system consisting of a spring, a mass, and a damping device (a device that dissipates the kinetic energy of the oscillator). Without the damping device, the mass will oscillate indefinitely. With the device, its motion gradually dies out.

Inside the cylindrical damping device there is a piston immersed in a fluid. This fluid resists the motion of the piston through it. This resistance can be controlled by varying the viscosity of the fluid or by otherwise varying its ability to bypass the piston.
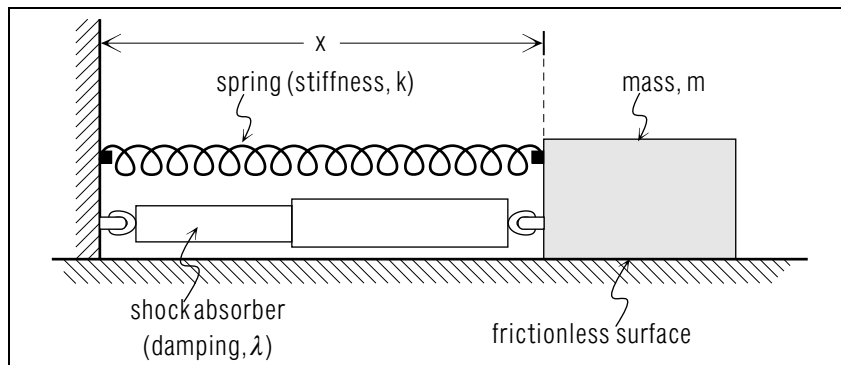
Figure 1. A simple damped oscillator.

1c. Varying the Resistance to Motion. If we gradually increase the ability of the damping device in Fig. 1 to resist motion, we get the series of curves shown in Fig. 2. Each curve shows the displacement of the oscillator from its position of static equilibrium as a function of time.

The first curve is with no damping. Succeeding curves are for increasing resistance of the damping device to motion, such as by making the fluid go from water to light oil to heavy oil to thick molasses (the quantity γ is a measure of this increasing resistance). Note that a point is reached beyond which there are *no* oscillations. This transition point, called the *critical damping* point, occurs at $\gamma = 2\pi f_0$.

1d. Prospectus. In the rest of this module we will first treat each forces acting on the mass separately, then combine the forces. This means we will review the restoring force that produces simple harmonic motion, then introduce the damping force and see what it does to the motion. In the process we will make quantitative the ideas we have discussed. This will provide an understanding of a wide variety of cases and devices. It will also lay the ground work for considering damped *driven* oscillatory motion, wherein energy is continually fed into the system as well as being dissipated from it. The addition of the driving force is treated elsewhere.[a]

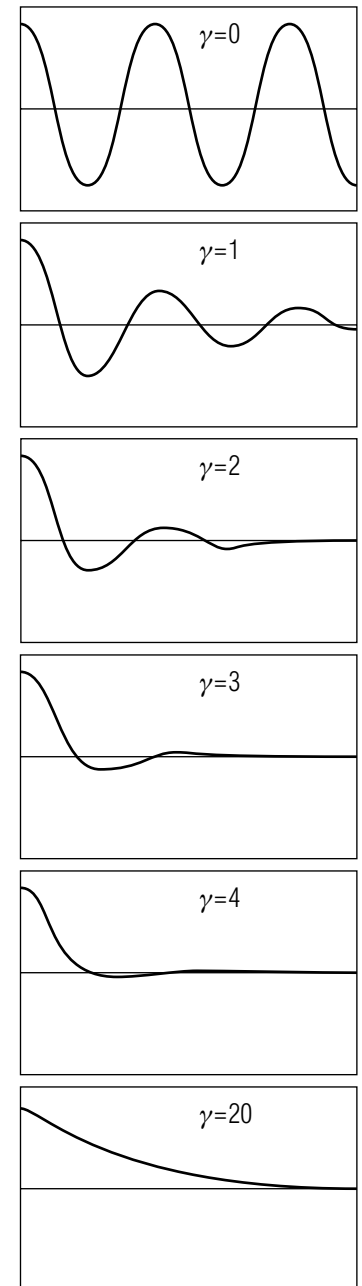[a]See "Damped Driven Oscillatory Motion," (MISN-0-30).

Fig. 2. (see text).

2. The Forces

2a. The Restoring Force. The restoring force is the force that, at all times, accelerates the oscillator back toward the position of static equilibrium. If the displacement is sufficiently small for the case at hand, the restoring force will be linear to a good approximation so we write it:

$$F = -kx. \qquad (1)$$

A force that obeys Eq. (1), with an interpretation of k appropriate for that force, is called a "Hooke's law force." If it is the only force acting on a mass, the mass will undergo Simple Harmonic Motion (hereafter referred to as "SHM"):

$$x(t) = x(0) \cos(\omega_0 t + \alpha). \qquad (2)$$

▷ Substitute Eqs. (2) and (1) into $F = ma$ and show that the equation is satisfied if the constant ω_0 is equal to the correct combination of m and k. *Help: [S-1]* [1]

2b. The Damping Force. A "damping force" is one that, acting by itself, (smoothly) stops motion. For example, friction acting on a horizontally coasting bicycle or car is a damping force because it gradually brings the vehicle to a halt.

A damping force must oppose the velocity in order that the resultant acceleration is a deceleration. This means that, mathematically, the damping force has the opposite sign to the velocity. The damping force cannot be a constant since that would decrease the velocity through zero and then cause it to increase in the direction of the force, whereas the damping force (by itself) is to make the motion stop and then stay stopped. If the damping is sufficiently small for the case at hand, it is likely to be linear to a good approximation:

$$F = -\lambda v. \qquad (3)$$

If this is the only force acting on a mass, the velocity will decrease to zero exponentially in time:

$$v(t) = v(0) \exp(-\gamma t). \qquad (4)$$

▷ Substitute Eqs. (4) and Eq. (3) into $F = ma$ and show that the equation is satisfied if the constant γ is equal to the correct combination of λ and

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[1] If you need help in this, see sequence [S-1] in this module's *Special Assistance Supplement*.

m. *Help: [S-2]* Since the differential equation is second order and homogeneous, this solution is the unique solution (apart from transformations to mathematically equivalent functions).

▷ Visualize Eq. (4) as a graph and think of how the curve on it changes as the damping constant λ and mass m are increased and decreased. *Help: [S-3]*

▷ Integrate Eq. (3) to give: $x(t) = x(0) - [v(0)/\gamma] \exp(-\gamma t)$.

2c. The Two Forces Together. A linearly-damped linearly-restored oscillator is one in which the force acting on the mass is the sum of Eqs. (1) and (2):

$$F = -kx - \lambda v. \qquad (5)$$

In this case we will write the solution differently depending on whether λ is smaller, equal to, or larger than the combination $2\sqrt{mk}$. These three cases are referred to as "underdamped," "critically damped," and "overdamped." We could write a single solution for all three cases but it would involve complex variables. To avoid that, we separate the three cases:

$$\begin{aligned} \text{underdamped:} \quad & \lambda < 2\sqrt{mk} \\ \text{critically damped:} \quad & \lambda = 2\sqrt{mk} \\ \text{overdamped:} \quad & \lambda > 2\sqrt{mk} \end{aligned}$$

▷ Note that the progression in the names matches the progression in the size of the damping constant λ.

3. The Damped Oscillator Solutions

3a. The Underdamped Solution. For the underdamped case, the solution to $F = ma$ can be written in the form:

$$x(t) = A e^{-\gamma t} \cos(\omega t + \alpha). \qquad (6)$$

Here A and α must be fixed by two known values of x, or its derivatives, at one or two specific times. For example, A and α could be fixed by knowing x at two different times or by knowing x at a specific time and the velocity v at a specific time.

▷ Substitute Eq. (6) and Eq. (5) into $F = ma$ and show that the equation is satisfied if the constants γ and ω are equal to the correct combinations

of k, λ, and m: *Help: [S-4]*

$$\gamma = \lambda/(2m)\,, \tag{7}$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2}\,, \tag{8}$$

where we define the undamped frequency, ω_0, as usual:

$$\omega_0 = \sqrt{k/m}\,. \tag{9}$$

Note that the condition for the underdamped case can be written: $\lambda < \omega_0$.

▷ Sketch a graph of Eq. (6). Note that the frequency of the oscillations, ω, remains constant in time but that the amplitude of the oscillations dies out exponentially. To make the sketch, first construct positive and negative decreasing exponential functions on the graph using dashed lines. The result should be a graph that is symmetric under reflections about the time axis. Now draw the constant-frequency exponentially-dying oscillations inside the dashed-lines "envelope." *Help: [S-5]*

3b. The Overdamped Solution. For the overdamped case, $\gamma > \omega_0$, the solution can be written in the form:

$$x(t) = A'\, e^{-(\gamma+\omega')t} + B'\, e^{-(\gamma-\omega')t}\,. \tag{10}$$

where γ and ω_0 are as before and:

$$\omega' = \sqrt{\gamma^2 - \omega_0^2}\,. \tag{11}$$

As in the underdamped case, A' and B' must be fixed by two known values of $x(t)$ and/or its derivatives. Eq. (10) can be shown to be mathematically equivalent to Eq. (6).[2]

▷ Sketch a graph of Eq. (10). Note that the curve is just the sum of two decaying exponentials. To the inexperienced eye, the sketch will look more or less like a single decaying exponential.

▷ Make sketches of what happens to the curve as you increase the damping constant to make it farther and farther from "critical" damping.

3c. The Critically Damped Solution. For the critically damped case, $\omega = \omega' = 0$. Using this, we can create a simpler form of Eq. (6) or

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[2]If you are interested, see this module's *Appendix*.

(10) by letting ω or ω' approach zero in either of those equations.[3] This gives:

$$x(t) = (A'' + B''t)\, e^{-\gamma t}\,. \tag{12}$$

Again, A'' and B'' must be fixed by two known values of $x(t)$ and/or its derivatives.

▷ Sketch a graph of Eq. (12).

3d. The Transition. In Fig. 2 we have plotted $x(t)$, Eqs. (6) and (10), for increasing values of the damping constant λ. The critically damped case is not shown but corresponds to: $\gamma = \omega_0 = 2\pi f_0$. We have chosen the constants such that $\alpha = 0$.

Think of the system shown in the figure as the wheel of a car that has just gone over a sharp bump. If the shock absorber is worn out and hence inoperative, the damping is zero and the wheel oscillates up and down as in the first figure. If the shock absorber is almost worn out, so the damping is small, the wheel responds as in the next figure. If the shock absorber produces too much damping, the wheel takes a long time to come back to the equilibrium position, as shown in the last figure. Obviously, critical damping is close to what one wants. It is also obvious that the shock absorbers must be matched to the weight of the car and to the stiffness of its springs.

Notes for Those Interested

a. The Equivalence of Eqs. (10) and (6). To see that the two equations are equivalent, note that $\omega^2 = -\omega^2$ so $\omega' = i\omega$ where $i = \sqrt{-1}$. Make this substitution and finish the conversion using $e^{ix} = \cos x + i \sin x$. *Help: [S-6]*

b. Derivation of Eq. (12). To show that Eq. (12) is the limit of Eq. (6) as ω approaches zero, use the approximation: $e^x \approx 1 + x$ for small

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[3]If you are interested in the details, see this module's *Appendix*.

x. Help: [S-7]

c. Uniqueness of the solution. A fundamental existence theorem in mathematics guarantees that there is only one solution to $F = ma$ for a force that is linear in x, v, and/or a, providing that force contains no higher derivatives and providing the solution contains two independently adjustable constants. Thus any solution we find that satisfies $F = ma$, with the restrictions noted, is the only solution. That is why we we can show that the three forms are equivalent.

Although the solution is unique in a mathematical sense, it can also be thought of as a two-parameter family of solutions, the two parameters being the two adjustable constants. For example, consider the plotted curve for an underdamped case, showing oscillations inside a decaying-exponential envelope. Assume the value zero for the phase α so the curve is at its highest value at the origin. Now start continuously increasing α and watch as the curve changes shape, going through complete oscillational cycles down and up the vertical axis at the origin as α continues to increase. Of course the constant A can also be varied and together the two variations produce the two-parameter family of solutions for fixed λ, k, and m.

PROBLEM SUPPLEMENT

1. A meter needle is to be designed so that its amplitude of vibration will drop by 99.0% in three oscillations, hence be unobservable after that. If the frequency of oscillation of the needle is 2.00 per second, determine the design damping constant γ.

2. An engineering student observing the diving board at the swimming pool notices that the vibrating board's amplitude changes from 14.0 cm to 8.00 cm in 0.800 seconds. Determine the damping constant γ of the board.

Brief Answers:

1. $\gamma = 3.07 \,/\mathrm{s}$

2. $\gamma = 0.700 \,/\mathrm{s}$

SPECIAL ASSISTANCE SUPPLEMENT

S-1　　　*(from TX-2a)*

$F = ma$ becomes:
$-kx = m(d^2x/dt^2)x = m[-x(0)\omega_0^2 \cos{(\omega_0 t + \alpha)}] = -m\omega_0^2 x$.

We set $\omega_0^2 = k/m$, substitute it in on the right hand side, and the equation is satisfied.

S-2　　　*(from TX-2b)*

$f = ma$ becomes:
$-\lambda v = m(d/dt)v = m[-v(0)\gamma \exp{(-\gamma t)}] = -m\gamma v$.

We set $\gamma = \lambda/m$, substitute it in on the right hand side, and the equation is satisfied.

S-3　　　*(from TX-2b)*

As γ increases the curve falls off more sharply.

S-4　　　*(from TX-3a)*

$x = A \exp{-\gamma t} \cos{(\omega t + \alpha)}$
Differentiate x to get:
$v = -\gamma A \exp{-\gamma t} \cos{(\omega t + \alpha)} - \omega A \exp{-\gamma t} \sin{(\omega t + \alpha)}$
hence:
$v = -\gamma x - \omega A \gamma \exp{-\gamma t} \sin{(\omega t + \alpha)}$
hence:
$\omega A \exp{-\gamma t} \sin{(\omega t + \alpha)} = -v - \gamma x$.
Now differentiate v to get:
$a = -\gamma v + \omega A \gamma \exp{-\gamma t} \sin{(\omega t + \alpha)} - \omega^2 A \exp{-\gamma t} \cos{(\omega t + \alpha)}$
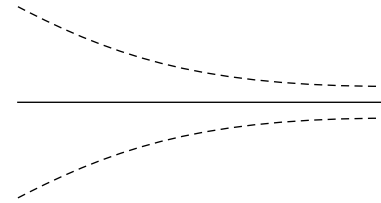or:
$a = -(2\gamma)v - (\gamma^2 + \omega^2)x$.
Then $f = ma$ becomes:
$-kx - \lambda v = ma = -(2\gamma m)v - [m(\gamma^2 + \omega^2)]x$.
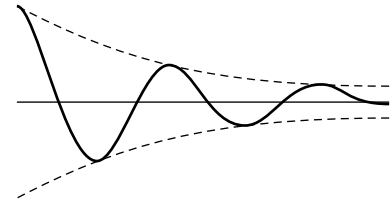We set $\gamma = \lambda/(2m)$ and $\omega^2 = \omega_0^2 - \gamma^2$ and the equation is satisfied (with $\omega_0^2 = k/m$).

S-5　　　*(from TX-3b)*

Step 1:　　　　　　　　　　Step 2:

S-6　　　*(from Appendix-a)*

Compare the definitions of ω and ω' and you will see that: $\omega' = i\omega$. We substitute that into Eq. (9) and find:
$x = A'e^{-\gamma t - i\omega t} + B'e^{-\gamma t + i\omega t}$.
Now $e^{a+b} = e^a e^b$ hence:
$x = e^{-\gamma t}[A'e^{-i\omega t} + B'e^{+i\omega t}]$.
We now set $A' = Ae^{-i\alpha/2}$ and $B' = Ae^{+i\alpha/2}$
so we get: $x = Ae^{-\gamma t} \cos{(\omega t + \alpha)}$ which is just Eq. (6).
Note: $\cos x = (e^{ix} + e^{-ix})/2$.

S-7　　　*(from Appendix-b)*

Write Eq. (9) as:
$x = e^{-\gamma t}(A'e^{-\omega' t} + B'e^{+\omega' t})$.
We now make ω' small so $e^{\pm\omega' t} \approx (1 \pm \omega' t)$.
Substituting this, we get:
$x = e^{-\gamma t}[A'(1 - \omega' t) + B'(1 + \omega' t)]$,
or:
$x = e^{-\gamma t}[(A' + B') + t(\omega' B' - \omega' A')]$.
We now set $A'' = A' + B'$ and $B'' = \omega'(B' - A')$ so we get:
$x = e^{-\gamma t}(A'' + B'' t)$, which is just Eq. (12).

MODEL EXAM

1. See Output Skills (Knowledge) K1-K4 on this module's *ID Sheet*.

2. An engineering student observing the diving board at the swimming pool notices that the vibrating board's amplitude changes from 14.0 cm to 8.00 cm in 0.800 seconds. Find the damping constant γ of the board.

Brief Answers:

1. See the text and Special Assistance Supplement.

2. See Problem 2 in the Problem Supplement.