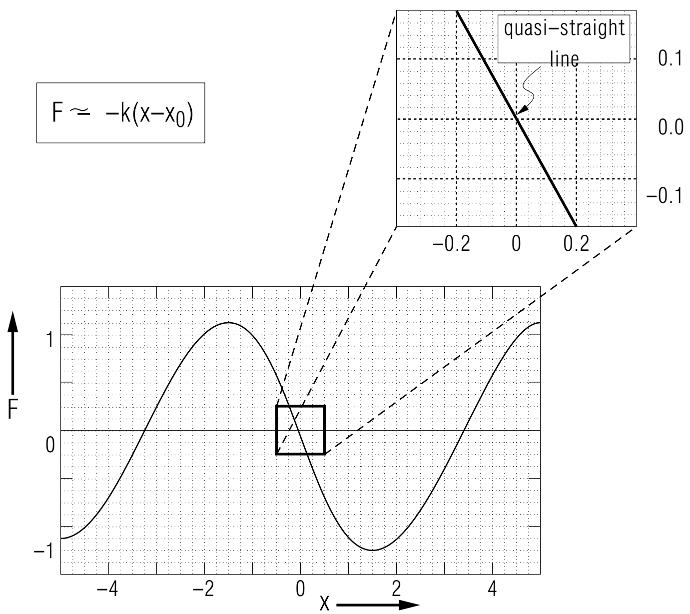
MISN-0-28

SMALL OSCILLATIONS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SMALL OSCILLATIONS

by

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Title: **Small Oscillations**

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Input Skills:

1. Vocabulary: simple harmonic motion, linear restoring force, frequency, displacement, amplitude, phase (MISN-0-26).

Output Skills (Knowledge):

K1. Vocabulary: small oscillation, point of stable equilibrium, linear approximation.

Output Skills (Problem Solving):

S1. Given a force or potential energy function, show the small oscillation solution's: (i) center point; (ii) force constant; and (iii) frequency. The solution presentation should include: intermediate steps, graphs, and answer checks.

Post-Options:

1. "Interatomic Binding and Lattice Energy of Crystalline Solids" (MISN-0-334).
2. "Diatomic Molecules: Properties from Rotation-Vibration Spectra" (MISN-0-217).

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1. Introduction

1a. Approximating Real-Force Oscillations. The purpose of this lesson is to help you develop skill in using Simple Harmonic Motion (SHM) as an approximation to general oscillatory motions. Strictly speaking, SHM applies only to oscillations produced by a linear restoring force: $F = -k(x - x_0)$. Other forces are non-linear functions of position x but we often approximate them with the SHM force because: (1) the SHM solutions can be easily obtained in terms of well-known mathematical functions; (2) most oscillations are well-approximated by SHM when their amplitudes, driving forces and damping are small; (3) the SHM approximation often gives us useful insights; and (4) we often need some approximation with which to check the computer program used to obtain the actual motion.[1]

1b. Most Forces Look Like SHM Over Small Displacements. The reason SHM is a good approximation for "small" oscillations is because almost any force function looks rather like the linear SHM force if you examine it in a small enough region around the center point of its oscillations (see Figure 1). Thus if the amplitude of the oscillation is small enough, for the particular force you have at hand, then SHM will be a good enough approximation to the actual motion and no further work need be done. You must decide whether the SHM solution is good enough for your particular application.

1c. Checking Computer Codes. If computer solutions are needed, the computer code should be checked by comparing small-amplitude numerical solutions, produced by the code, to the corresponding SHM solutions. One does this by running the program with the input constants temporarily set at a succession of "small oscillation" amplitudes. One plots the resulting frequencies versus the log of the amplitude and extrapolates the curve to zero amplitude. This "zero amplitude" frequency should agree exactly with the SHM frequency. If they do agree, one then puts the amplitude back at its physical value and runs the computer program again to get the physical answer.

[1]For descriptions of the terms "driving forces" and "damping" see *Damped Driven Oscillations and Mechanical Resonances* (MISN-0-31).

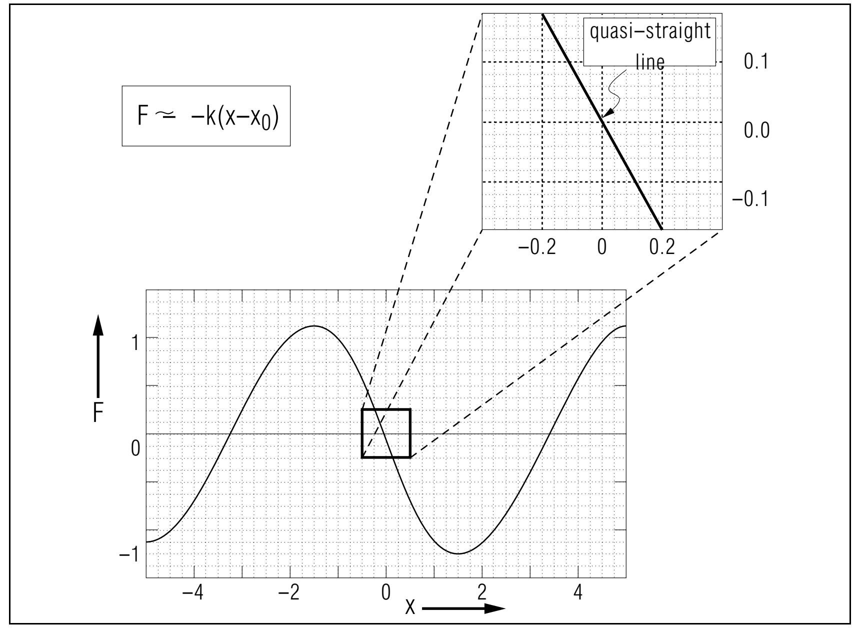


Figure 1. A non-linear function usually looks rather straight over a small enough region.

1d. Real Force In, Approximate Motion Out. In this module we show you how to start with: (1) some particular restoring force: and (2) the mass of the oscillator driven by that force, and from that input produce the particular SHM solution which the real oscillation approaches as the amplitude approaches zero. The "solution" consists of the SHM approximation to: (1) the oscillator's frequency (hence also its period); and (2) the center point of its oscillations. Using these, you can immediately write down the oscillator's position as a function of time.

2. Example: a Pendulum

2a. The Physical Oscillator. Our illustrative example is the simple pendulum. This consists of a rod which is pivoted at its upper end and which has a weight, called a "bob," attached at the rod's lower end (see Figure 2). The rod is assumed to be weightless and of length ℓ. The bob is assumed to be the size of a mathematical point and is assumed to have mass m. Our pendulum is assumed to be swinging back and forth, under

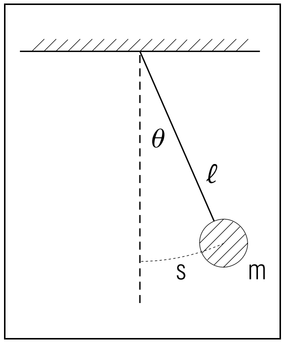


Figure 2. A simple pendulum.

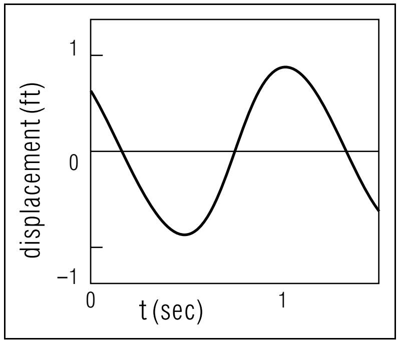


Figure 3. Position vs. time for the pendulum of Fig. 2.

the influence of gravity, without friction. We can think of it as a slightly idealized version of the pendulum in a Grandmother's or Grandfather's clock.

2b. The Position Oscillates. The position of the bob oscillates as the bob travels back and forth along its circular-arc trajectory. This position is denoted s in Figure 2, and it is taken as zero at the lowest point on the trajectory. The position $s(t)$ oscillates: as time increases it goes smoothly from positive values to negative ones to positive ones to negative ones, etc. (see Figure 3).

2c. Finding the Appropriate SHM. We produce the approximating SHM solution, the one which the real force's oscillations approach as amplitude becomes small, by carrying out these three steps:

1. Obtain the net force acting on the oscillator. For our pendulum, we start by noting that two different objects exert forces on the bob, which is the mass that oscillates: the earth through gravity, and the rod through its point of contact with the bob. Show that the resultant force is tangential to the trajectory and its component in the direction of increasing s is: $F(s) = -mg\,sin(s/\ell)$ (this is the only component there is). *Help: [S-2]* [2]

2. Construct a linear force whose value and slope agree with those of the true force at the point where the true force descends through zero (the point of zero displacement for the oscillations). This

[2] For help, see sequence S-2 in this module's Special Assistance Supplement.

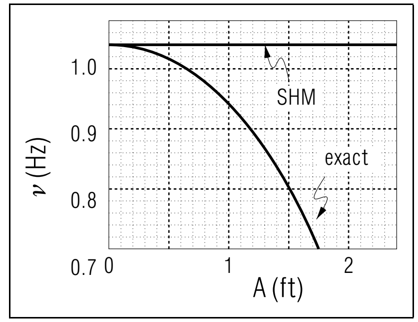


Figure 4. Exact and SHM frequencies, model as in Figure 2.

means that we must first find the place where the true force descends through zero, then find the slope of the true force there. For our pendulum, the point where the true force (the force on the bob) descends through zero is seen by inspection to be $s = 0$. The slope of the force there is: $dF/ds(0) = -mg/\ell$. The linear force which has that value and slope at that point is (check this in your head): $F(s) = -mgs/\ell$. For sufficiently small displacements around the origin, the linear force and the true force appear indistinguishable to the eye (see Figure 1).

3. Calculate the SHM frequency for the linear approximating force. For our pendulum, the force constant of the approximating SHM is: $k = mg/\ell$. This gives the SHM frequency: $\nu_0 = (1/2\pi)\omega$ where $\omega = \sqrt{g/\ell}$. If the pendulum amplitude is small, its frequency is close to this SHM frequency.

2d. Comparing the Exact and SHM Values. In Figure 4 we compare the SHM frequency to the true frequency as a function of the amplitude of the oscillations. The SHM frequency is independent of the SHM oscillator's amplitude so it is plotted as a constant. However, the pendulum's frequency varies with the pendulum's amplitude. The figure shows that if the pendulum swings to 60°, the SHM prediction for the frequency is off by about 6%, but if the swing is to 30° then SHM is off by only about 1.5%![3]

In Figure 5 we compare the positions as functions of time for various amplitudes. Note that the angle of swing must get up to 125° (picture that swing!) before the curve no longer looks like an SHM sine curve. Note also that the frequency decreases (so the period increases in Figure

[3] See "Small Oscillations Revisited," MISN-0-32.

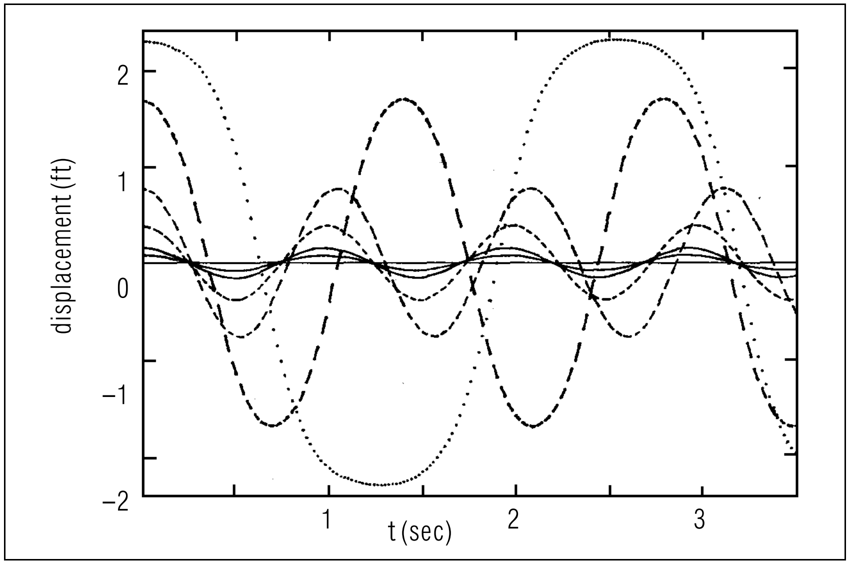


Figure 5. Position vs. time for various amplitudes, model as in Fig. 2.

5) as the amplitude increases.

3. Variations

3a. Summary. We must often deal with these variations: (1) the center point of the oscillations is not at the coordinate origin; (2) the potential energy is specified instead of the force; and (3) rather than a fixed force acting on a single mass, we have two masses exerting mutual forces on each other as they each oscillate.

3b. Oscillation Not About the Origin. It is often inconvenient to put the origin of the coordinate system at the place where the force descends through zero. The coordinate position where the force descends through zero is called the "Point of Stable Equilibrium" and it is denoted by adding a subscript "zero" to the coordinate symbol. We will usually refer to the Point of Stable Equilibrium as the "PSE."[4] For example, if the position of the oscillator is labeled x then its PSE is labeled x_0. If

[4] For a discussion of classes of equilibrium points, see "Static Equilibrium" (MISN-0-6). The abbreviation "PSE," used here, is not in common use: there is no commonly used abbreviation.

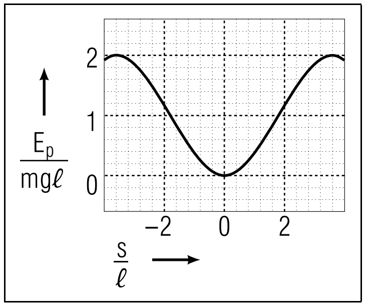


Figure 6. Potential energy vs. displacement, model as in Figure 2.

the position of the oscillator is labeled s then its PSE is labeled s_0.

3c. The Potential Energy is Specified. Often a potential energy function is specified rather than a force function.[5] Since $F(x) = -dE_p(x)/dx$, x_0 is the point where the slope of E_p, E_p', is ascending through zero. The force constant is then the second derivative: $k = E_p''(x_0)$. When the potential energy function $E_p(x)$ is plotted as a function of x, it shows a minimum at the PSE (see Figure 6).

3d. Two Mutually-Interacting Bodies. If two masses are oscillating due to equal but opposite forces each exerts on each other, the parameters we have been using must be reinterpreted. Now x is the (variable) distance between the two objects, $F(x)$ is the force each body exerts on the other, and x_0 is the equilibrium separation of the two objects, the separation at which each exerts zero force on the other. When x is larger than x_0, the two masses are attracted toward each other; when x is smaller than x_0, they are repelled from each other. The force constant for the system is defined as before: $k = -dF/dx(x_0)$. The two objects' oscillations will be identical but 180° out of phase (they both head away from each other, then they both turn around and head toward each other, then they turn around and head away from each other, etc., each oscillating about its own end of x_0).[6] The SHM frequency is then: $\nu = (1/2\pi)\sqrt{k/\mu}$, where the system's "reduced mass" μ is given by: $\mu \equiv m_1 m_2/(m_1 + m_2)$.[7]

[5] For a discussion of how to find the force function corresponding to a given potential energy function see "Potential Energy Curves, Force, and Motion" (MISN-0-22).

[6] Note that the total momentum of the two-body system is always zero.

[7] This is derived in: "Two-Body Kinematics and Dynamics" (MISN-0-45).

4. Solution Steps

1. If E_p is given, find F.
2. Find the PSE.
3. Find k.
4. Determine the appropriate mass: take m or calculate μ.
5. Find ν.

5. The Point of Stable Equilibrium

5a. Formal Solution. One way for finding the zeros of a function is formally, which means setting the function equal to zero and then solving that equation formally. The solutions are the function's zeros. Then you must find which of those zeros are PSE's by examining the sign of F' at each of them. If there is more than one PSE, your particular application must specify which PSE to focus on.

For a linear force, find the PSE without writing anything down. For example, in the function $F(x) = -a(x - b)$, the point $x = b$ is obviously a PSE. Here is another linear form for you to try: $F(x) = -a^2x + b^2$. *Help: [S-4]*

For a quadratic force, use the quadratic root equation. For example, for $F = 3 - 16x + 5x^2$, use the quadratic root formula to show that: (1) there is a PSE at $x_0 = 0.2$ *Help: [S-10]*; and (2) there is another root at $x = 3$ but this one is not a PSE because the force is ascending, not descending, through zero there. This latter point is called a "point of unstable equilibrium." The word "equilibrium" is used because an object placed at rest at this point will not move, and the word "unstable" is used because any small displacement of the object from the point will cause the object to be accelerated away from that point.

There are now good computer programs, employing artificial intelligence techniques, that can produce most of the known formal solutions for roots. The most used programs at this time are MACSYMA, REDUCE, and MATHEMATICA.

5b. Numerical Techniques. When a formal solution cannot be found, or sometimes when it cannot be found easily, we use numerical techniques.

Sometimes we use computer programs or calculators (like the HP - 41) that incorporate a whole bag of numerical zero-finding tricks. If you have access to a zero-finding calculator or computer, try it on the quadratic force given above.

Sometimes we use numerical interpolation, mainly on a hand calculator: this always works, albeit slowly. Try numerical interpolation on your own calculator, on the force $F(x) = -\cos x$. Calculate values of $F(x)$ for various values of x, finding a region where F changes from positive to negative. Single out the adjacent pair of points between which F changes sign. Use linear interpolation on this pair of points to predict the point where F is zero. Calculate the actual value of F at that point and repeat the interpolation process as many times as you wish. *Help: [S-15]*

5c. Graphical Method. For the graphical method, use your hand calculator and graph paper, or a computer, to plot the force function. Note the point where the force descends through zero. Increase your accuracy by "blowing up" the region around that zero on a new plot. Repeat this enlarging process until you have located the PSE to the desired accuracy. You can try this on the quadratic function given above.

6. The Force Constant

6a. Formal Method. Once the PSE has been found, by whatever means, the force constant k can usually be found formally; that is, by taking the first derivative of the force and then evaluating it at the PSE. For example,

$$F = (3 - 16\,x\,\mathrm{m}^{-1} + 5\,x^2\,\mathrm{m}^{-2})\,\mathrm{N}\,,$$

can be differentiated at its PSE, $x_0 = 0.2\,\mathrm{m}$, to obtain: $F' = -14\,\mathrm{N/m}$. Then the force constant can be seen by inspection: $k = 14\,\mathrm{N/m}$. If you don't know a particular function's derivatives, look them up in a table of derivatives[8] or use an appropriate formal-math computer program.

6b. Finite Difference and Graphical Methods. If the derivative cannot easily be taken formally, it can be measured as the slope of the force on a graph, or it can be evaluated using finite difference equations. The graphical method is usually not practical for taking second derivatives, so finite difference equations are better when the potential energy function is given instead of the force function.

[8]See the Appendix or, for example, *A Short Table of Integrals*, B. D. Peirce, Ginn and Co., Boston (1929).

The relevant finite difference equations are:[9]

$$f'(x) \simeq \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta}\,;$$

$$f''(x) \simeq \frac{f(x+\Delta) - 2f(x) + f(x-\Delta)}{\Delta^2}\,.$$

In the above formulas x is the PSE and Δ is a small distance. The distance Δ should be made sufficiently small so that making it smaller results in no significant improvement in the value of k (for our present purposes, 2-3 significant digits). Do not make Δ so small that k's first 2-3 digits are affected by calculator error. You can practice on: $F = -5\,x^{-4} + 20\,x^{-6}$. *Help: [S-3]*

Note that use of the graphical or finite difference methods as a check provides a virtually fail-safe method for locating all errors inadvertently made while taking formal derivatives.

Here is a function on which you can easily practice all three methods (formal, graphical, and numerical): $F = -\cos x$. *Help: [S-1]*

Acknowledgments

For their generous assistance with this module, I would like to thank: Jack Hetherington, George Bertsch, Tom Kaplan, Gerry Pollack, Jon Pumplin, Jules Kovacs, Robert Spence, and Jerry Cowen. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **Point of Stable Equilibrium (PSE)**: for a particular force, a space-point where any small displacement produces a return force (Sections 3b and 5).
- **Small Oscillation**: an oscillation whose motion is simple harmonic within tolerable limits.
- **Linear Approximation (to a particular mathematical function in the neighborhood of a specific point)**: the straight line which has the same value and slope as the designated function at the designated point.

[9] See "Taylor's Series for the Expansion of a Function About a Point" (MISN-0-4).

Table of Derivatives

Combinations of functions.

1. $\dfrac{d}{dx}\,(f(u)) = \dfrac{df(u)}{du} \cdot \dfrac{du}{dx}$
2. $\dfrac{d}{dx}\,(f \pm g) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$
3. $\dfrac{d}{dx}\,(fg) = f\,\dfrac{dg}{dx} + g\,\dfrac{df}{dx}$
4. $\dfrac{d}{dx}\,(f/g) = \left(g\,\dfrac{df}{dx} - f\,\dfrac{dg}{dx}\right) / \,g^2$

Specific Functions.

5. $\dfrac{d}{dx}\,(x^n) = n\,x^{n-1}$
6. $\dfrac{d}{dx}\,(e^x) = e^x$
7. $\dfrac{d}{dx}\,(\ell n\ x) = 1\,/\,x$
8. $\dfrac{d}{dx}\,(\sin x) = \cos x$
9. $\dfrac{d}{dx}\,(\cos x) = -\sin x$

When the argument contains a constant. For example, in

$$\frac{d}{dx}\,(\sin(ax))\,,$$

first use formula #1 ($u = ax$ in this example), then use the formula appropriate to the function (#8 in this example).

PROBLEM SUPPLEMENT

HOW TO WORK THESE PROBLEMS. Each problem involves the exercise of eight skills. We suggest you practice the first skill across the first few problems, one after the other, until you get the hang of it. Then come back to the first problem and move on to the next skill. The answers are arranged to match that procedure. When you have mastered all eight skills, try working the rest of the problems straight through. Here are the skills:

a. Differentiate E_p and use that derivative to get the force function F.

b. Solve for the symbolic r_0, a zero of F.

c. Substitute numbers to find the numerical value of r_0.

d. Plot F around r_0; check value of r_0 and check descending linearity.

e. Differentiate F to get the symbolic $F'(r_0)$.

f. Substitute numbers to get $k = -F'(r_0)$.

g. Determine the appropriate mass for the frequency formula.

h. Calculate ν_0, the "frequency of small oscillations."

NOTE. The introductory paragraphs and the units that are used need not be understood to work these problems! Just plunge in and apply each of the eight skills, one after the other. Work problems 1-6 for practice in using the formal method for skills b-c. Work problems 7-9 for practice in using the numerical-graphical method for skills b-c.

1. **The Gravity-Body Segment Interaction and Stepping Frequency [General Interest, Biomechanics, Simple Pendula]**

   Let us see if the stepping frequency in normal human walking can be related to the pendulum-swing frequencies of the arm and the lower leg. First, we can calculate the arm and leg swing frequencies using body-segment mass distribution data. The computed frequencies can then be directly compared to the normal stepping frequency, or can be combined with observed distance-covered-per-step data to compare to normal walking speed.

The model outlined above treats the arm or lower leg as a pendulum that swings freely from its point of suspension. Of course each of them is, in reality, driven to some extent, but they should swing fairly freely in relaxed walking where the individual does not wish to exert much effort. Thus our results should be most valid for relaxed walking. Nevertheless, we will also be able to gain some insight into the way arm positions and motions change for running.

The arm and the lower leg are not simple pendula, but may be reasonably treated as such in a first approximation. We will denote the pendulum parameters by the symbols shown in Fig. 2 in the text.

In terms of the pendulum mass's arc-displacement s and the effective pendulum length ℓ, its potential energy function can be written (see Fig. 2): *Help: [S-2]*

$$E_p = mg\ell\,[1 - cos(s/\ell)] = 2mg\ell\, sin^2(s/2\ell)\,.$$

To treat the arm-plus-hand system as a simple pendulum, we take data from Hanavan's computer model of the adult male and Fischer's estimate of radius-of-gyration ratios for body segments.[10] The arm-plus-hand numbers we get for a 5 ft, 10 in, 150 lb person are: $\ell = 0.77$ ft, wt. $= 9.0$ lb. Thus: $m = 9.0\,\text{lb}/g$ where $g = 32\,\text{ft/s}^2$.

Note that Figures 1-4 in the text illustrate this arm-gravity problem.

Carry out the eight steps to find the frequency of small oscillations of the arm-hand system about the equilibrium position.

After you have completed the eight skills, carry out these steps to see how this problem relates to real life:

- Check the arm-swing frequency you just computed against your own stepping frequency (the frequency at which you take steps when you allow your arms to swing like pendulums).
- Use the arm-swing frequency you just calculated and 5 or 6 feet as the distance covered per step, or your own measured stepping distance, to compute your walking speed in ft/s or mi/hr.
- Demonstrate how runners shorten their arm-plus-hand value of ℓ as they pick up speed, and explain why they must shorten ℓ.

[10] See D. I. Miller and R. C. Nelson, *Biomechanics of Sport: A Research Approach*, Lea & Febiger, Philadelphia (1973), esp. p. 85, 98. For other details of stepping see "The Control of Walking," Keir Pearson, *Scientific American 235*, p. 72 (1976).

- Use the dependence of ν on ℓ to explain why a tall person has to exert a significant effort to keep in synchronous step with a short person who is walking normally.

2. **The Earth-Sun Interaction, Orbital Eccentricity [General Interest, Astrophysics, Meteorology, Geology].** The cycle of the earth-orbit's eccentricity, or "out of roundness," governs the time between the earth's short bursts of warm temperature between glacial periods, while the two shorter Milankovic cycles participate in determining the height and width of these pulses.[11] The earth-orbit eccentricity oscillates on a (roughly) 93,000 year cycle, so it will be about that amount of time before the earth has another "high" like the just-ending one during which our present civilization developed. The period of ice and drought now starting may well provide an interesting test of mankind's ability to fashion artificial climates.[12]

   The present eccentric motion of the earth can be graphically illustrated by plotting the current orbit parameters on a potential energy curve. Including both the centrifugal and gravitational potentials, we find for the earth-sun potential energy function:

$$E_p(r) = a(r^{-2} - br^{-1})$$

   where:

$$a = 2.65649 \times 10^{33}\,\mathrm{J}\,(\mathrm{AU})^2$$
$$b = 2\,(\mathrm{AU})^{-1}$$
$$\mathrm{AU} \equiv \text{one "Astronomical Unit" of distance} = 1.49647\ \times 10^{11}\,\mathrm{m}$$
$$\text{earth mass} = 1.345\ \times 10^{32}\,\mathrm{J}\,(\mathrm{AU})^2\,\mathrm{yr}^2$$

   and r is the earth-sun separation.

   Carry out the eight steps to find the frequency of small oscillations of the earth about its equilibrium distance from the sun.

3. **The 8n-8p Interaction in ^{16}O [Nuclear Physics].** The giant resonances are considered some of the most spectacular phenomena in nuclear physics, for several reasons. First, their coherent group-motions of nuclear particles are very different from the individual-particle motions implied by the hallowed shell model. Second, they are spectacular simply for their sheer size, sometimes comprising 90% or more of all the observed scattering for a given projectile and target excitation. Finally, they convey information about the nuclear force: for example, neutron excitation of the giant isovector dipole state could not occur if the two-body nuclear force was isospin independent.

   The giant dipole state in ^{16}O is due to a separation of the center-of-mass (CM) of the eight protons (8p) from that of the eight neutrons (8n). If we denote the distance between the two CM's by x, the separation potential energy can be written[13]

$$E_p(x) = -\,(a + bx^2 + cx^4)e^{-dx^2}$$
$$a = 277.21\,\mathrm{MeV}$$
$$b = 17.329\,MeV\,\mathrm{fm}^{-2}$$
$$c = 0.7825\,MeV\,\mathrm{fm}^{-4}$$
$$d = 0.16215\,\mathrm{fm}^{-2}.$$
$$m_{8\mathrm{p}} = 8.34\ \times 10^{-44}\,\mathrm{MeV\,s^2 fm^{-2}}$$
$$m_{8\mathrm{n}} = 8.35\ \times 10^{-44}\,\mathrm{MeV\,s^2 fm^{-2}}$$

   Carry out the eight steps to find the frequency of small oscillations of the (8p) CM and the (8n) CM about their positions of equilibrium separation.

4. **The Argon-Argon "6-12" Interaction [Solid State, Low Temperature Physics; Materials Science].** Solid argon has recently been studied as a model solid, one whose properties are theoretically tractable. It is a simple solid to study because each atom's electrons are tightly bound to it, resulting in each lattice site being occupied by an identical neutral argon atom. In contrast to ionic crystals, such a neutral-atom solid has no inter-atomic Coulombic interactions. The remaining interaction is thus weak and of short range, with the result that only nearest neighbors can interact significantly. This produces a tremendous simplification in deducing the properties of the solid from the pair-wise interatomic forces.

[11] J. D. Hays, John Imbrie, and N. J. Shackelton, "Variations in the Earth's Orbit: Pacemaker of the Ice Ages," *Science*, 194, 1121 (Dec. 10, 1976), esp. Fig. 9B, center, and *Science News*, 110, 356 (Dec. 4, 1975). For the 25,800 yr. cycle see "Vector Angular Momentum and Torque: Precession of the Equinoxes" (MISN-0-77). Also: Nigel Calder, *The Weather Machine*, Viking Press, NY (1976, Viking Compass Edition).

[12] See "Small Oscillations Revisited," MISN-0-32.

[13] G. Bertsch and K. Stricker, "On the Macroscopic Theory of the Giant Dipole State," *Phys. Rev. C* **13**, 1312 (1976) and private communication from G. Bertsch.

The interaction potential energy for two argon (Ar) atoms a distance r apart can be well-represented by the Lennard-Jones function:[14]

$$E_p(r) = -ar^{-6} + br^{-12}$$

$$a = 17.33\,\text{eV}\,\text{Å}^6$$

$$b = 2.923\,\times 10^4\,\text{eV}\,\text{Å}^{12}.$$

$$m_{Ar} = 4.135\,\times 10^{-27}\,\text{eV}\,\text{Å}^{-2}\,\text{s}^2$$

Carry out the eight steps to find the frequency of small oscillations of two Ar atoms their positions of equilibrium separation.

5. **The He-He "6-12" Interaction [Solid State, Liquid State, Quantum Crystal Physics].** Helium crystals have aroused interest recently because they are truly quantum crystals; the helium atoms of which the crystals are composed cannot be treated as classical particles. In fact, direct application of classical dynamics produces an erroneous unstable equilibrium point exactly at the observed lattice site of an atom, along with erroneous stable equilibrium points elsewhere. Thus the classical lattice treatment says that the observed stable lattice should be unstable, and it erroneously says that other lattice arrangements should be stable. In the correct quantum treatment, one begins with a potential energy function for pairs of atoms, treated as classical point-particle force centers, then obtains the potential energy function at any particular lattice site by summing over nearby quantum densities.

   The two-body potential energy function for two helium atoms a distance r apart is well approximated by[15]

$$E_p(r) = a\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

   where

$$a = 40.88\,\text{K}\,k_{\text{Boltzmann}}$$

$$\sigma = 2.556\,\text{Å}.$$

$$m_{He} = 4.808 \times 10^{-24}\,k_{\text{B}}\,\text{K}\,\text{Å}^{-2}\,\text{s}^2$$

   Carry out the eight steps to find the frequency of small oscillations of two He atoms about their positions of equilibrium separation.

[14] G. L. Pollack, "The Solid State of Rare Gases," *Rev. Mod. Phys.* **36**, 748 (1964).

[15] L. H. Nosanow, "Theory of Quantum Crystals," *Phys. Rev.* **146**, 120 (1966).

6. **The Electron-Proton Interaction ($\ell = 1$) [Atomic Physics].** The hydrogen atom is the simplest of all atoms and the only one for which the correct quantum equation can be solved formally. The input for the quantum Schrödinger equation is the classical electron-proton potential energy function.

   For any $\ell = 1$ state of atomic hydrogen, the electron(e)-proton(p) potential energy function is:

$$E_p(r) = ar^{-2} - br^{-1}$$

   where:

$$a = 0.0763\,\text{eV}\,\text{nm}^2$$

$$b = 1.440\,\text{eV}\,\text{nm}$$

$$m_p = 1.043\,\times 10^{-26}\,\text{eV}\,\text{nm}^{-2}\text{s}^2$$

$$m_e = 5.678\,\times 10^{-30}\,\text{eV}\,\text{nm}^{-2}\text{s}^2$$

   The first term is the "centrifugal repulsion" and the second term is the Coulombic e-p attraction.

   Carry out the eight steps to find the frequency of small oscillations of the electron and the proton in the $\ell = 1$ state of the hydrogen atom.

7. **The Reid Nucleon-Nucleon Potential [Nuclear Physics].** The "Reid soft-core potential" is an interaction sometimes used in nuclear calculations to describe the force between each pair of nucleons in a nucleus. Here is Reid's potential for the $^1\text{S}_0$ part of the force between a neutron (n) and a proton (p):[16]

$$E_p(r) = (-he^{-x} - Ae^{-4x} + Be^{-7x})/x$$

   where

$$x = 0.7\,\text{fm}^{-1}\,r$$

$$A = 1650.6\,\text{MeV}$$

$$B = 6484.2\,\text{MeV}$$

$$h = 10.463\,\text{MeV}.$$

$$m_p = 1.043\,\times 10^{-44}\,\text{MeV}\,\text{fm}^{-2}\text{s}^2$$

$$m_n = 1.044\,\times 10^{-44}\,\text{MeV}\,\text{fm}^{-2}\text{s}^2$$

   Carry out the eight steps to find the frequency of small oscillations of a neutron and a proton about their positions of equilibrium separation.

[16] R. V. Reid, Jr., "Local Phenomenological Nucleon-Nucleon Potentials," *Annals of Physics* (NY) **50**, 411 (1968).

8. **Ba-Ti-O "1-6-9" Interactions (Solid State Physics, Materials Science).** Barium titanate crystals ($BaTiO_3$) are of considerable commercial interest because they exhibit a region of ferroelectricity, with a relative dielectric constant of the order of ten thousand!

 Our picture of normal barium titanate is that of an ionic crystal in which the Ba, Ti, and O ions are regarded as point force centers. For any two of the Ba, Ti, or O ions, with charges e_1 and e_2 and a distance r apart, the interaction produces a potential energy represented by[17]

$$E_p(r) = e_1\, e_2\, r^{-1} - \mu\, r^{-6} + \lambda\, r^{-9}.$$

 The first term is the Coulombic interaction, the second is the Van der Waals attraction, and the third is a short range repulsion due mainly to the exclusion principle. The charges on the Ba, Ti, and O ions are $2e$, $4e$, and $-2e$ respectively, with $-e$ being the charge on the electron.

 For an interacting oxygen-ion-barium-ion pair,

$$\lambda_{\text{O - Ba}} = 6180\,\text{eV}\,\text{Å}^9$$

$$\mu_{\text{O - Ba}} = 101\,\text{eV}\,\text{Å}^6$$

$$e_1\, e_2 = -4e^2 = -57.6\,\text{eV}\,\text{Å}$$

$$m_{\text{O}} = 1.656 \times 10^{-27}\,\text{eV}\,\text{Å}^{-2}\text{s}^2$$

$$m_{\text{Ba}} = 1.421 \times 10^{-26}\,\text{eV}\,\text{Å}^{-2}\text{s}^2$$

 Carry out the eight steps to find the frequency of small oscillations of a barium ion and an oxygen ion about their equilibrium separation in $BaTiO_3$.

9. **The Quark-Antiquark "Gluon-Bag" Interaction ($\ell = 1$) [Elementary Particle, High Energy Physics].** Perhaps the most exciting experimental discovery in recent elementary particle physics is that of the ψ particles, each apparently consisting of a charmed quark (q) and a charmed antiquark ($\bar{q}$) held together in their bag by gluons. (The charm of the quark and antiquark add to zero, so the ψ has no charm.)

 A typical P-state potential energy function assumed for the q and $\bar{q}$ pair, separated by a distance r is[18]

$$E_p(r) = a\, r^{-2} - b\, r^{-1} + c\, r$$

[17] A. F. Devonshire, "Theory of Barium Titanate", *Phil. Mag.* **40**, 1040 (1949).

[18] E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan, *Phys. Rev. Lett.* **34**, 369 (1975).

 where

$$a = 0.097\,\text{GeV}\,\text{fm}^2$$

$$b = 0.20\,\text{GeV}\,\text{fm}$$

$$c = 5.0\,\text{GeV}\,\text{fm}^{-1}$$

 and:

$$m_q = m_{\bar{q}} = 1.78 \times 10^{-47}\,\text{GeV}\,\text{fm}^{-2}\text{s}^2.$$

 Carry out the eight steps to find the frequency of small oscillations of the $q\bar{q}$ pair about their positions of equilibrium separation.

Brief Answers:

1a. $F = -(W)\sin(s/\ell)$, where W is the weight

2a. $F = 2ar^{-3} - abr^{-2}$

3a. $F = [x(-2ad + 2b) + x^3(4c - 2bd) + x^5(-2cd)]e^{-dx^2}$

4a. $F = -6ar^{-7} + 12br^{-13}$

5a. $F = a(12\sigma^{12}r^{-13} - 6\sigma^6 r^{-7})$

6a. $F = 2ar^{-3} - br^{-2}$

7a. $F = -(1/r)(he^{-ar} + 4Ae^{-4ar} - 7Be^{-7ar})$

$-(1/ar^2)(he^{-ar} + Ae^{-4ar} - Be^{-7ar})$; $a \equiv x/r$

8a. $F = e_1 e_2 r^{-2} - 6\mu r^{-7} + 9\lambda r^{-10}$

9a. $F = 2ar^{-3} - br^{-2} - c$

1b. $s_0 = 0$

2b. $r_0 = 2/b$

3b. $x_0 = 0$

4b. $r_0 = (2b/a)^{1/6}$

5b. $r_0 = \sigma(2)^{1/6}$

6b. $r_0 = 2a/b$

7b. graphical or numerical

8b. graphical or numerical

9b. graphical or numerical or solve cubic equation to get:

$$r_0 = \frac{(-3)2^{2/3}b + [54ac^{1/2} + (2916a^2c + 108b^3)^{1/2}]^{2/3}}{(3)2^{1/3}c^{1/2}[54ac^{1/2} + (2916a^2c + 108b^3)^{1/2}]^{1/3}}$$

1c. $s_0 = 0$

2c. $r_0 = 1\,\mathrm{AU}$

3c. $x_0 = 0$

4c. $r_0 = 3.872666\,\text{Å} \approx 3.87\,\text{Å}$

5c. $r_0 = 2.86901\,\text{Å} \approx 2.87\,\text{Å}$

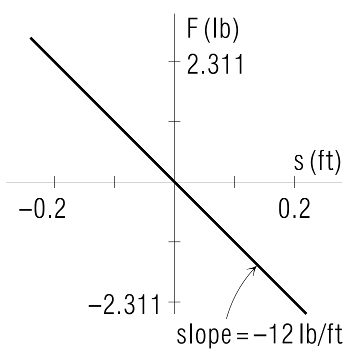
6c. $r_0 = 0.10597222\,\mathrm{nm} \approx 0.106\,\mathrm{nm}$

7c. $r_0 = 0.844818\,\mathrm{fm} \approx 0.845\,\mathrm{fm}$

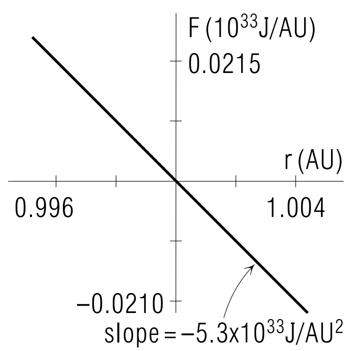
8c. $r_0 = 2.31836\,\text{Å} \approx 2.32\,\text{Å}$

9c. $r_0 = 0.299353627\,\mathrm{fm} \approx 0.299\,\mathrm{fm}$

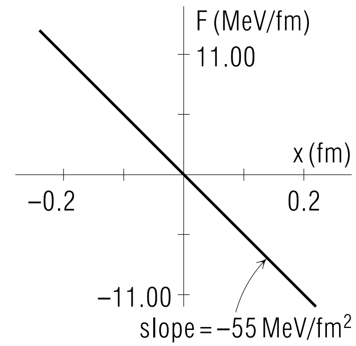
1d.



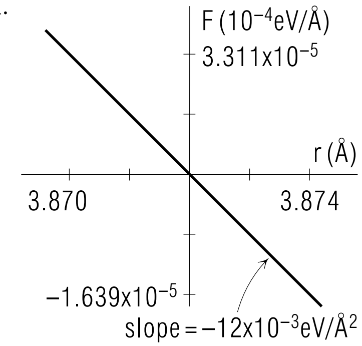
2d.



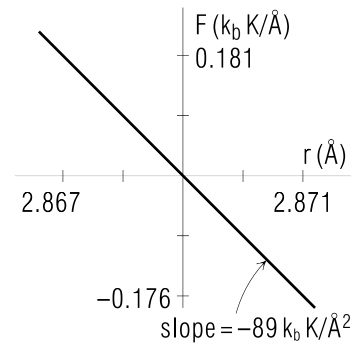
3d.



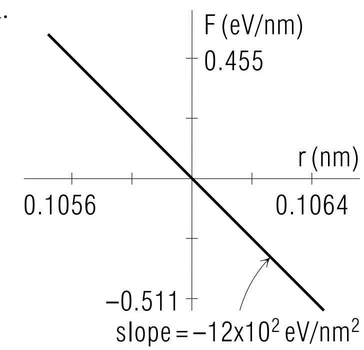
4d.



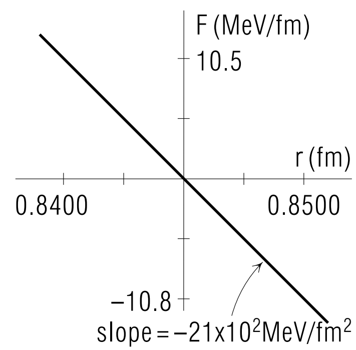
5d.



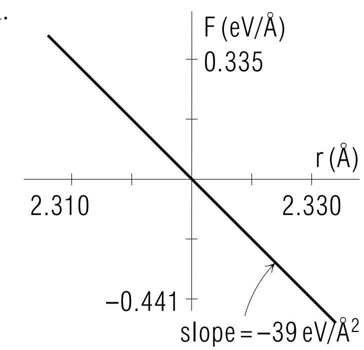
6d.



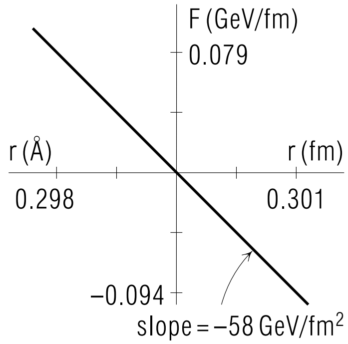
7d.



8d.



9d.



1e. $F'(s_0) = -W/\ell$

2e. $F'(r_0) = -ab^4/8$

3e. $F'(x_0) = 2(b - ad)$

4e. $F'(r_0) = -18(a^2/b)(a/2b)^{1/3}$

5e. $F'(r_0) = -(a/\sigma^2)(18)2^{-1/3}$

6e. $F'(r_0) = -b^4/(8a^3)$

7e. advice: do it numerically or graphically

8e. $F'(r_0) = -2e_1e_2r_0^{-3} + 42\mu r_0^{-8} - 90\lambda r_0^{-11}$

9e. $F'(r_0) = -6ar_0^{-4} + 2br_0 - 3$

1f. $k = 11.688\,\mathrm{lb/ft}$

2f. $k = 5.31 \times 10^{33}\,\mathrm{J/(AU)^2}$

3f. $k = 55.2\,\mathrm{MeV/fm^2}$

4f. $k = 0.01233\,\mathrm{eV/Å^2}$

5f. $k = 89.4\,k_\mathrm{B}\,\mathrm{K/Å^2}$

6f. $k = 1210\,\mathrm{eV/nm^2}$

7f. $k = 2130\,\mathrm{MeV/fm^2}$

8f. $k = 39.1\,\mathrm{eV/Å^2}$

9f. $k = 57.6\,\mathrm{GeV/fm^2}$

1g. $m = 0.28125\,\mathrm{lb\,ft^{-1}s^2}$

2g. $m = 1.339 \times 10^{47}\,\mathrm{J\,(AU)^{-2}s^2}$

3g. $m =$ reduced mass $= 4.172 \times 10^{-44}\,\mathrm{MeV\,fm^{-2}s^2}$

4g. $m =$ reduced mass $= 20.625 \times 10^{-28}\,\mathrm{eV\,Å^{-2}s^2}$

5g. $m =$ reduced mass $= 2.404 \times 10^{-24}\,k_\mathrm{B}\,\mathrm{Å^{-2}s^2}$

6g. $m =$ reduced mass $= 5.675 \times 10^{-32}\,\mathrm{eV\,Å^{-2}s^2}$

7g. $m =$ reduced mass $= 52.17 \times 10^{-46}\,\mathrm{MeV\,fm^{-2}s^2}$

8g. $m =$ reduced mass $= 14.83 \times 10^{-28}\,\mathrm{eV\,Å^{-2}s^2}$

9g. $m =$ reduced mass $= 8.90 \times 10^{-48}\,\mathrm{GeV\,fm^{-2}s^2}$

1h. $\nu_0 = 1.03\,\mathrm{Hz}$

2h. $\nu_0 = 3.17 \times 10^{-8}\,\mathrm{Hz}$

3h. $\nu_0 = 5.79 \times 10^{21}\,\mathrm{Hz}$

4h. $\nu_0 = 3.89 \times 10^{11}\,\mathrm{Hz}$

5h. $\nu_0 = 9.71 \times 10^{11}\,\mathrm{Hz}$

6h. $\nu_0 = 2.32 \times 10^{15}\,\mathrm{Hz}$

7h. $\nu_0 = 1.02 \times 10^{23}\,\mathrm{Hz}$

8h. $\nu_0 = 2.58 \times 10^{13}\,\mathrm{Hz}$

9h. $\nu_0 = 4.05 \times 10^{23}\,\mathrm{Hz}$

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-6b)*

$F(x) = -\cos x$

formal: $k = F'(x_0)$

$$\text{PSE is at } x_0 = 3\pi/2: \; F(3\pi/2) = -\cos(3\pi/2) = 0$$

$$F' = \sin x: \quad F'(3\pi/2) = \sin(3\pi/2) = -1.$$

graphical: Use upper graph in Fig. 1: take a Δx of 0.10 (say), find the corresponding ΔF, and calculate:

$$F' = \frac{\Delta F}{\Delta x} = \frac{-0.10}{0.10} = -1.0.$$

Another way to see it: The F and x scales are the same and the line is at a 45° angle to the horizontal. Thus the magnitude of the slope is unity: its sign is obviously negative.

numerical: (using a hand calculator)

x	F	Differences
4.70	0.01239	
		> -0.01000
4.71	0.00239	
		> -0.01000
4.72	-0.00761	

$$\Longrightarrow F' = \frac{-0.01000}{0.01} = -1.000$$

S-2 *(from TX-2c and PS-problem P1))*

$E_p = mgh$: h is height above the horizontal plane containing the bottom point of the swing (a convenient choice for $E_p = 0$). Then $h = \ell\,(1 - \cos\theta)$ by simple trig. (*Help: [S-7]*) The rest involves differentiating sines and cosines. *Help: [S-5]*

S-3 *(from TX-6b)*

$F = -5x^{-4} + 20x^{-6}$. Use hand calculator to evaluate it:

	x	F	Comments
	1.9	4.14×10^{-2}	Magnitudes at $+\Delta$ and $-\Delta$
center	2.0	1.00×10^{-10}	are not sufficiently
	2.1	-2.39×10^{-2}	equal. Try smaller Δ.
	1.99	3.21×10^{-3}	
	2.00	1.00×10^{-10}	Almost. Try smaller.
	2.01	-3.04×10^{-3}	
	1.999	3.13×10^{-4}	
	2.000	1.00×10^{-10}	O.K.; $F' \simeq -0.3125$
	2.001	-3.12×10^{-4}	

$$\textit{check}: \; -\frac{5}{16} = -0.3125$$

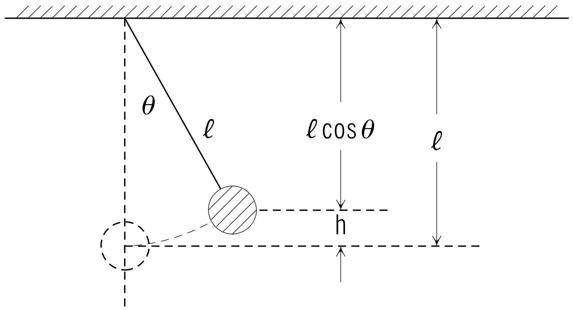
S-4 *(from TX-5a)*

$0 = -a^2 x_0 + b^2 \;\Longrightarrow\; x_0 = b^2/a^2$
$F' = -a^2 \;\Longrightarrow\; k = a^2$

S-5 *(from Help: [S-2])*

$E_p = mg\ell[1 - \cos(s/\ell)]$
$F = -E'_p = -dE_p/ds = -mg\sin(s/\ell)$
$F' = \text{etc.}$

S-7 *(from [S-2])*



S-10 *(from TX-5a)*

PSE (x_0) is where F is zero:

$$0 = 3 - 16x_0 + 5x_0^2.$$

We use the quadratic root equation:

$$\text{if: } ax^2 + bx + c = 0$$

$$\text{then: } x = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

$$\text{and find: } x_0 = 0.2, 3.$$

Then get the sign of the slope at each point (0.2, 3) by differentiation (*Help: [S-13]*), by sketching the function, or by numerical evaluation at two nearby points: see "numerical" part of *Help: [S-2]*.

S-12 *(from TX-6a)*

If you can't work this problem, first work those in Sect. 4c. Then you should have no trouble. The PSE is at; $x_0 = 0.2\,\text{cm}$. The force constant is: $k = 14\,\text{N/cm}$. The mass is: $m = 1\,\text{gm} = 10^{-3}\,\text{kg} = 10^{-3}\,\text{N}\,\text{m}^{-1}\text{s}^2$. *Help: [S-16]*

$$\text{Thus:}\quad \nu = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{1/2} = \frac{1}{2\pi}\left(\frac{14\,\text{N}\,\text{cm}^{-1}}{10^{-3}\,\text{N}\,\text{m}^{-1}\text{s}^2}\right)^{1/2}$$
$$= \frac{1}{2\pi}\left(\frac{14\,\text{cm}^{-1}10^2\,\text{cm/m}}{10^{-3}\,\text{m}^{-1}\,\text{s}^2}\right)^{1/2}$$
$$= 1.88 \times 102\,\text{s}^{-1} = 188\,\text{Hz}$$

S-13 *(from [S-10])*

$$F = 3 - 16x + 5x^2$$

$$F'(x) = -16 + 10x$$

$$F'(0.2) = -14 \ (< 0 \text{ so is PSE})$$

$$F'(3) = +14 \ (> 0 \text{ so is not PSE})$$

S-15 *(from TX-5b)*

x	F	Linear interpolation
3	.9900	
4	.6536	$x_0 \simeq 4 + \dfrac{.6536}{.6536 + .2837} = 4.697$
5	-.2837	
4.6	.1122	
4.7	.0124	$x_0 \simeq 4.7 + \dfrac{.0124}{.0124 + .0875} = 4.7124$
4.8	-.0875	
4.7124	.0000	good enough!

S-16 *(from [S-12])*

For background on this conversion of units, see "Newton's Second Law of Motion" (MISN-0-14).

MODEL EXAM

$$\text{reduced mass: } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

1. Define...(see Output Skill K1 in this module's *ID Sheet*).

2. The present eccentric motion of the earth can be graphically illustrated by plotting the current orbit parameters on a potential energy curve. Including both the centrifugal and gravitational potentials, we find for the earth-sun potential energy function:

$$E_p(r) = a(r^{-2} - br^{-1})$$

where:

$$a = 2.65649 \times 10^{33}\,\mathrm{J}\,(\mathrm{AU})^2$$
$$b = 2\,(\mathrm{AU})^{-1}$$

$$1\,\mathrm{AU} \equiv \text{one "Astronomical Unit of distance"} = 1.49647 \times 10^{11}\,\mathrm{m}$$
$$\text{earth mass} = 1.345 \times 10^{32}\,\mathrm{J}\,(\mathrm{AU})^{-2}\,\mathrm{yr}^2$$

and r is the earth-sun separation.

Determine the frequency of small oscillations of the earth about its equilibrium distance from the sun, through explicit use of these steps, labeled this way in your answers: (a) symbolic F; (b) symbolic r_0; (c) numerical r_0; (d) graphical check; (e) symbolic F'; (f) numerical k; (g) appropriate mass; and (h) numerical ν_0.

As always, show all of your reasoning.

Brief Answers:

1. See this module's text.

2. See this module's *Problem Supplement*, Problem 2.