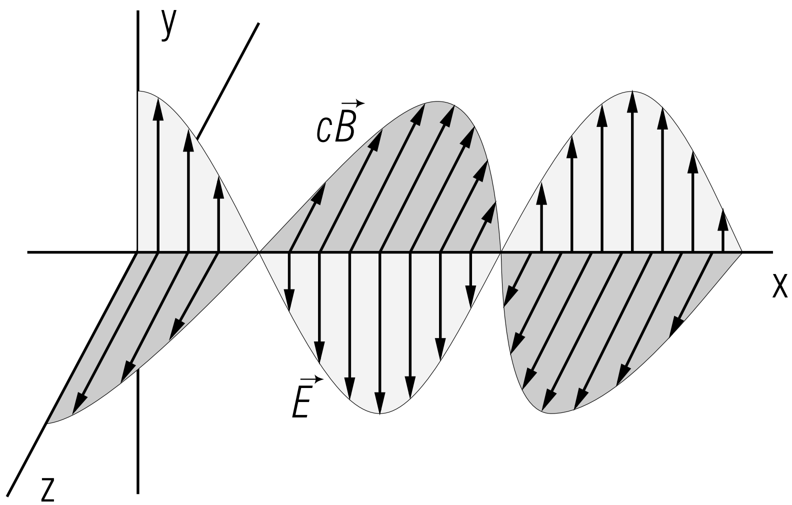
PHYSNET

MISN-0-210

ELECTROMAGNETIC WAVES FROM MAXWELL'S EQUATIONS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ELECTROMAGNETIC WAVES FROM MAXWELL'S EQUATIONS

by

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Title: **Electromagnetic Waves from Maxwell's Equations**

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Input Skills:

1. Vocabulary: charge density (MISN-0-147), current density (MISN-0-118), displacement (MISN-0-25), sound waves, wave frequency, wavelength, waves on strings, wave speed (MISN-0-202), wave equation (MISN-0-201).
2. Take derivatives of transcendental functions (MISN-0-1).
3. Take scalar and vector products of vectors using Cartesian unit vectors (MISN-0-2).

Output Skills (Knowledge):

K1. Vocabulary: propagation (of a wave), polarization (direction of), plane-polarized (wave), monochromatic (wave).

K2. Given Maxwell's Equations, the "curl-curl" vector identity, and the definitions of the gradient, divergence, and curl operators, derive the wave equations for electric and magnetic field vectors at chargeless currentless space-points.

Output Skills (Rule Application):

R1. Given the definitions of the gradient, divergence, and curl operators, verify that a given electromagnetic wave, consisting of coupled electric and magnetic waves, satisfies Maxwell's Equations.

R2. Given the direction of polarization, direction of propagation, frequency and amplitude of a monochromatic plane-polarized electromagnetic wave, write down the electric and magnetic fields in vector form. Sketch the situation.

Post-Options:

1. "Energy and Momentum in Electromagnetic Waves," (MISN-0-211).
2. "Brewster's Law and Polarization," (MISN-0-225).

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1. Introduction

1a. Background. The transport of energy through mechanical systems via the collective motion of the particles that make up the system is a familiar phenomenon, spectacularly demonstrated, for example, when the voice of a soprano shatters a glass across the room from her. This energy is carried by the "displacement" waves that can be made to propagate through the system (the "displacement" referring to the displacement from equilibrium of the particles of the system or of the pressure in the gas).

1b. Waves in Space. It is also possible for electric and magnetic fields to propagate as waves in empty space, the electric and magnetic field vectors playing the same role in electromagnetic waves as the transverse displacement of the particles of a string do in waves along a stretched string, or the pressure displacement associated with the propagation of sound waves in air. The important difference is that there is no medium through which this electromagnetic wave propagates, although perhaps one could say the "medium" is the vacuum! Electric and magnetic fields may exist in space without a material medium being present, and if they vary in space and time in the appropriate way, the spatial variation will propagate as a wave, transporting energy. This module deals with the propagation of energy through a vacuum via electromagnetic disturbances whose space and time variation satisfy the conditions for wave propagation.

1c. Vector Derivatives. To deal with electromagnetic waves in space it is far easier to use Maxwell's equations in *derivative* form than in *integral* form. In electricity and magnetism we are dealing with scalar fields like the charge density ρ and vector fields like $\vec{E}$, and we will find that we must deal with three kinds of derivative operators: the "gradient" operator that operates on a scalar field and produces a vector one, the "divergence" operator that operates on a vector field and produces a scalar one, and the "curl" operator that operates on a vector field and produces

another vector one.

1d. Partial Derivatives. When taking derivatives of field quantities we generally use "partial derivatives," denoted ∂, to remind us that the spatial coordinates are not functions of time. That is, whereas a particle has a single value of, say, x, at any particular time, a field has values at a continuum of values of x. Otherwise, partial derivatives are like ordinary derivatives:

$$\frac{\partial}{\partial x}\left(x^3y^4z^5\right) = 3x^2y^4z^5\,,$$

$$\frac{\partial}{\partial x}(\sin 3x\cos y) = 3\cos 3x\cos y\,.$$

2. Vector Derivatives

2a. The Gradient Operator. The gradient operator operates on a scalar function and produces a vector function that is the steepest "uphill" slope at any point where the vector function is evaluated. That is, the gradient of a function, evaluated at some space-point, points in the direction that is most steeply "up hill" in that function at that point. The magnitude of the gradient is the value of the slope in the "steepest ascent" direction at that space-point. As an example, suppose the gradient of a field scalar field f is the vector field $\vec{g}$. In Cartesian coordinates this is:

$$\vec{g} = \vec{\nabla} f \equiv \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}\,. \qquad (1)$$

$\triangleright$ Show that $\vec{\nabla}(x^3y^4) = (3x^2y^4)\hat{x} + (4x^3y^3)\hat{y}$.

2b. The Divergence Operator. The "divergence" operator operates on a vector function, say $\vec{g}(x,y,z)$, to give a scalar function $f(x,y,z)$:

$$f = \vec{\nabla}\cdot\vec{g} \equiv \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}\,. \qquad (2)$$

The divergence of a vector function, evaluated at some space-point, gives the extent to which the function has a source or sink at that point. For example, the divergence of an electric field gives the charge density at that point (positive charges are sources for the field, negative charges are sinks).

$\triangleright$ Show that: $\vec{\nabla}\cdot\left(x^3y^4z^5\right)\hat{z} = 5x^3y^4z^4$.

2c. The Curl Operator. The "curl" operator operates on a vector function, say $\vec{g}(x, y, z)$, to give another vector function:

$$\vec{\nabla} \times \vec{g} \equiv \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\hat{z} + \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\hat{x} + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\hat{y}\,. \quad (3)$$

The curl of a function, evaluated at some space-point, gives the greatest "circulation" at that point, where by "circulation" one means the line integral of the function around a loop of infinitesimal radius. The direction of the curl is normal to the plane of the loop with the greatest line integral.[1]

▷ Show that: $\vec{\nabla} \times \left(x^3y^4z^5\right)\hat{z} = \left(4x^3y^3z^5\right)\hat{x} - \left(3x^2y^4z^5\right)\hat{y}$.

3. Maxwell's Equations

Here are the famous Maxwell's Equations in differential form:

$$\vec{\nabla} \cdot \vec{E} = 4\pi k_e \rho\,, \quad (4)$$

$$\vec{\nabla} \cdot \vec{B} = 0\,, \quad (5)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \quad (6)$$

$$\vec{\nabla} \times \vec{B} = 4\pi k_m j + c^{-2}\frac{\partial \vec{E}}{\partial t}\,. \quad (7)$$

Gauss's law is the integral form of Eq. (4). Ampere's law is the integral form of Eq. (7) for the case where the electric field does not vary with time. The Ampere-Laplace-Biot-Savart law is derived from a combination of Eqs. (5) and (7), also for the case where the electric field does not vary with time. The Faraday-Henry law of magnetic induction is the integral form of Eq. (6).

4. Electromagnetic Waves

4a. No Charge, No Current; Waves. For the case where there is no charge or current at a point in space, it is easy to show that waves can exist there. We set $\rho = 0$ and $j = 0$ in Eqs. (4)-(7), then take the time derivative of both sides of Eqs. (6) and (7): *Help: [S-3]*

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}\,,$$

[1]For another way to remember the definition of the curl, see Appendix A.

$$\frac{\partial^2 \vec{B}}{\partial t^2} = -\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t}\,.$$

We now put Eqs. (6) and (7) into the right sides of the above two equations to get:

$$-\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \times (\vec{\nabla} \times \vec{E})\,, \quad (8)$$

$$-\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{\nabla} \times (\vec{\nabla} \times \vec{B})\,, \quad (9)$$

To further reduce the above equations, we make use of the identity:[2]

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}\,. \quad (10)$$

where $\vec{A}$ is any vector field and:

$$\nabla^2 \equiv \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.$$

Finally, using that identity and Eqs. (4) and (5) for our chargeless case, we get these two *wave equations* for waves traveling with velocity c:

$$\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B}\,, \quad (11)$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E}\,. \quad (12)$$

4b. Plane-Polarized Monochromatic Waves. We can write down solutions to the wave equations, Eqs. (11) and (12), for the case where the field vectors lie entirely in a plane and where the solutions contain only one frequency. Of course one must show that the solutions we write down really are solutions by substituting them into Eqs. (11) and (12) and showing that those equations are satisfied. For the electric field vector the plane-polarized monochromatic solution is:

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)\,, \quad (13)$$

[2]This identity is proved in Appendix B. Physicists often remember the rule,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\,,$$

as the words "BAC minus CAB" along with the positions of the parentheses. However, with $\vec{A} \equiv \vec{B} \equiv \vec{\nabla}$, we must rearrange the order on the right side of the equation to keep the operators to the left of the functions they operate upon. Thus we wind up with Eq. (10).

where the *direction* of $\vec{k}$ gives the direction of propagation of the electric wave and the *magnitude* of $\vec{k}$ is 2π divided by the wave's wavelength and is related to the wave's frequency through its velocity:[3]

$$k = \frac{2\pi}{\lambda}; \qquad \omega = 2\pi f; \qquad f = \frac{1}{T} = \frac{c}{\lambda}\,,$$

where ω is the wave's angular frequency, f is its frequency, T is its period, c is its speed, and λ is its wavelength.

$\triangleright$ Show that Eq. (13) satisfies Eq. (12) by direct substitution on both sides of Eq. (12). *Help: [S-2]*

Now with Eq. (13), Eq. (6) becomes:

$$\vec{\nabla} \times \vec{E} = -(\vec{k} \times \vec{E}_0)\sin(\vec{k}\cdot\vec{r} - \omega t)\,,$$

but this equals $(\partial/\partial t)\vec{B}$ which can only be true if the arguments in the cosine functions match:

$$\vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)\,. \tag{14}$$

We put the solutions, Eqs. (13) and (14), into Eqs. (11) and (12) and find:

$$\vec{B}_0 = \frac{1}{\omega}(\vec{k} \times \vec{E}_0) = \frac{1}{c}(\hat{k} \times \vec{E}_0)\,,$$

so:

$$\begin{aligned} \hat{E}_0 \times \hat{B}_0 &= \hat{k} \\ \vec{E}_0 \cdot \vec{B}_0 &= 0 \\ B_0 &= \frac{1}{c}E_0\,. \end{aligned} \tag{15}$$

$\triangleright$ Show that the picture on the cover of this module requires all three of Eqs. (15).

4c. Production by a Radio Transmitter. A vertical radio transmitter tower is a good example of a device that produces a plane-polarized monochromatic wave (the frequency of the wave is the frequency to which you set the dial in order to receive the wave). A large current is sent up and down the vertical tower, as a sine wave with a single frequency.

$\triangleright$ Suppose we look at the tower during the part of the cycle when the current is moving upward. Use Ampere's law to show that the magnetic field to the right of the antenna is as given on this module's cover. Then use Eqs. (15) tell you the direction of the electric field and the direction of propagation of the wave.

[3]These are general properties of waves: see "The Wave Equation," (MISN-0-201).

4d. How The Waves Manifest Themselves. Depending on its frequency, an electromagnetic wave may be a radio or television wave coming through the air to your receiver, or it could be an X-ray, or a gamma ray from a radioactive decay, or a ray of light of a particular color. These objects are all *identical* waves except for their frequencies.

Acknowledgments

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Glossary

- **propagation**: motion.
- **polarization, direction of**: in a plane-polarized electromagnetic wave, the direction of the electric field vector.
- **plane-polarization**: in an electromagnetic wave, the condition in which the electric field vector always lies in the same plane (in contrast to, say, circular polarization where the electric field vector rotates around the axis of propagation).
- **monochromatic**: in an electromagnetic wave, the condition of a wave having a single frequency, a single wavelength (in contrast to being a mixture of different wavelengths). In a more sophisticated view, it means that there is only one Fourier component.

A. The Curl as a Determinant

Recall that the cross-product of two vectors $\vec{C}$ and $\vec{D}$ can be written as the determinant

$$\vec{C} \times \vec{D} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ C_x & C_y & C_z \\ D_x & D_y & D_z \end{vmatrix}$$

The x-component of vector $\vec{C}\times\vec{D}$ is the term in the expanded determinant which is proportional to $\hat{x}$:

$$(\vec{\nabla}\times\vec{D})_x = \frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z}$$

The other two components of the curl of D are thus:

$$(\vec{\nabla}\times\vec{D})_y = \frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x}$$

$$(\vec{\nabla}\times\vec{D})_z = \frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y}$$

B. Proof of a Vector Identity

We here prove:

$$\vec{A}\times(\vec{B}\times\vec{C}) = -(\vec{A}\cdot\vec{B})\vec{C} + \vec{B}(\vec{A}\cdot\vec{C})\,.$$

We only need to prove the identity for one component since the others will follow by cycling the subscripts. So we take the x-component of the left side:

$$\begin{aligned}
&[\vec{A}\times(\vec{B}\times\vec{C})]_x\\
&\quad = A_y(\vec{B}\times\vec{C})_z - A_z(\vec{B}\times\vec{C})_y\\
&\quad = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\\
&\quad = -(A_yB_y + A_zB_z)C_x + B_x(A_yC_y + A_zC_z)\\
&\quad = -(A_xB_x + A_yB_y + A_zB_z)C_x + B_x(A_yC_y + A_zC_z + A_xC_x)\\
&\quad = -(\vec{A}\cdot\vec{B})C_x + B_x(\vec{A}\cdot\vec{C})
\end{aligned}$$

hence:

$$\vec{A}\times(\vec{B}\times\vec{C}) = -(\vec{A}\cdot\vec{B})\vec{C} + \vec{B}(\vec{A}\cdot\vec{C})$$

so substituting $\vec{\nabla}$ for $\vec{A}$ and $\vec{B}$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{C}) = -(\vec{\nabla}\cdot\vec{\nabla})\vec{C} + \vec{\nabla}(\vec{\nabla}\cdot\vec{C})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{C}) = -\nabla^2\vec{C} + \vec{\nabla}(\vec{\nabla}\cdot\vec{C})\,,$$

and the identity is proved.

PROBLEM SUPPLEMENT

▷ **Warning**: First make sure you have done the first five (out of the six) problems scattered through the text, marked like this warning. If you skip any one of them, you will probably not be prepared for even the first problem below.

Note: Problems 4-7 also occur in this module's *Model Exam.*

1. The electric field of a plane electromagnetic wave in vacuum is represented by:

   $E_x = 0\,,$

   $E_y = 0.50\,(\mathrm{N/C})\cos\left[2.09\,\mathrm{m}^{-1}(x-ct)\right]\,,$

   $E_z = 0\,.$

   a. Determine the wavelength, frequency, polarization, and propagation vector of the wave. *Help: [S-4]*

   b. Determine the components of the wave's magnetic field. *Help: [S-1]*

2. Solve (a) and (b) of Problem 1 for the wave represented by:

   $E_x = 0\,,$

   $E_y = 0.50\,(\mathrm{N/C})\cos\left[0.419\,\mathrm{m}^{-1}(x-ct)\right]\,,$

   $E_z = 0.50\,(\mathrm{N/C})\cos\left[0.419\,\mathrm{m}^{-1}(x-ct)\right]\,.$

3. Determine the components of the $\vec{E}$- and $\vec{B}$-fields which describe the following electromagnetic waves that propagate along the positive x-axis:

   a. A wave whose plane of $\vec{E}$-vibration makes an angles of 45° with the positive y- and z-axes.

   b. A wave whose plane of $\vec{E}$-vibration makes an angle of 120° with the positive y axis and an angle of 30.0° with the positive z-axis.

4. Given these electric and magnetic fields:

$$E_x = E_0 \cos \frac{2\pi}{\lambda}(y + ct), \quad E_y = 0, \quad E_z = 0,$$
$$B_x = 0, \quad B_y = 0, \quad B_z = 0.$$

   a. Determine whether or not these fields satisfy the wave equations. [D]

   b. Determine whether or not these fields satisfy Maxwell's Equations. [B]

   c. If your answer to (a) and (b) is yes, what relationship must exist between the E and B amplitudes? [A]

5. Given these electric and magnetic fields:

$$E_x = 0, \quad E_y = E_0 \sin[2\pi\nu \left(\frac{x}{c} - t\right)], \quad E_z = 0,$$
$$B_x = 0, \quad B_y = 0, \quad B_z = B_0 \sin[2\pi\nu \left(\frac{x}{c} - t\right)].$$

   (a), (b), (c): Repeat Problem 1 using the above components.

   Answers: (a) [G], (b) [E], and (c) [H].

6. With:

   $E_x = E_0 \cos(kz - \omega t)$, $E_y = E_0 \cos(kz - \omega t)$, $E_z = 0$,

   write down the space-time dependence of the components of the magnetic field that will result in an electromagnetic wave that satisfies both the wave equations and Maxwell's equations. [C]

7. A plane-polarized monochromatic electromagnetic wave of frequency ν has the electric field polarized in the z-direction and the wave propagates in the negative y-direction. Determine the components of $\vec{E}$ and $\vec{B}$ that satisfy Maxwell's equations and the wave equations. [F]

Brief Answers:

1. a. 3.01 m, 1.00×10^8 Hz, polarized in the $\hat{y}$ direction, $\vec{k} = 2.09\,\mathrm{m}^{-1}\hat{x}$.

   b. $B_x = 0\,,$
   $B_y = 0\,,$
   $B_z = 0.17 \times 10^{-8}\,\mathrm{T}\cos\left[2.09\,\mathrm{m}^{-1}(x - ct)\right]\,.$

2 a. 15.0 m, 2.00×10^7 Hz, polarized in the y-z plane, $\vec{k} = 0.419\,\mathrm{m}^{-1}\hat{x}\,.$

   b. $B_x = 0\,,$
   $B_y = -0.17 \times 10^{-8}\,\mathrm{T}\cos[0.419\,\mathrm{m}^{-1}(x - ct)]\,,$
   $B_z = +0.17 \times 10^{-8}\,\mathrm{T}\cos[0.419\,\mathrm{m}^{-1}(x - ct)]\,,$

3 a. $E_x = 0,$
   $E_y = +0.707 E_0 \cos(kx - \omega t).$
   $E_z = +0.707 E_0 \cos(kx - \omega t).$
   $B_x = 0,$
   $B_y = -0.707(E_0/c)\cos(kx - \omega t).$
   $B_z = +0.707(E_0/c)\cos(kx - \omega t).$

   b. $E_x = 0,$
   $E_y = -0.500 E_0 \cos(kx - \omega t),$
   $E_z = +0.866 E_0 \cos(kx - \omega t).$
   $B_x = 0,$
   $B_y = -0.866(E_0/c)\cos(kx - \omega t).$
   $B_z = -0.500(E_0/c)\cos(kx - \omega t),$

A. No such wave exists.

B. No.

C. $B_x = -\frac{E_0}{c}\cos(kz - \omega t),$

   $B_y = +\frac{E_0}{c}\cos(kz - \omega t),$

   $B_z = 0.$

D. Yes.

E. Yes.

F. $E_x = 0,$

$E_y = 0,$

$E_z = +E_0 \cos\left[2\pi\nu\left(\frac{y}{c} + t\right)\right],$

$B_x = -\frac{E_0}{c} \cos\left[2\pi\nu\left(\frac{y}{c} + t\right)\right],$

$B_y = 0,$

$B_z = 0.$

G. Yes.

H. $B_0 = E_0/c$.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 1)*

For electromagnetic waves, the three important directions are: $\hat{k}$, the direction of propagation; $\hat{E}$, the direction of the electric field; and $\hat{B}$, the direction of the magnetic field. Any two of these may be known in a problem and we must find the third. Since these three are mutually perpendicular, they obey this cyclic rule:

$$\hat{E} \times \hat{B} = \hat{k}\,,$$

$$\hat{k} \times \hat{E} = \hat{B}\,,$$

$$\hat{B} \times \hat{k} = \hat{E}\,,$$

where each line is obtained from the one above it by cycling the vectors one place to the right.

S-2 *(from TX-4b)*

Recall that $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$. Then:

$$\vec{\nabla}_x(\vec{k} \cdot \vec{r}) = \frac{\partial}{\partial x}(\vec{k} \cdot \vec{r}) = k_x\,.$$

and similarly for $\vec{\nabla}_y$ and $\vec{\nabla}_z$. Then:

$$\vec{\nabla}(\vec{k} \cdot \vec{r}) = \vec{k}\,.$$

This means that:

$$\vec{\nabla} \cos(\vec{k} \cdot \vec{r} - \omega t) = -\vec{k} \sin(\vec{k} \cdot \vec{r} - \omega t)\,.$$

Now you fill in the remaining steps to get:

$$\nabla^2 \vec{E} = -k^2 \vec{E}\,.$$

and do a similar job on the other side of the wave equation.

S-3 *(from TX-4a)*

The various partial derivatives, $(\partial/\partial t)$, $(\partial/\partial x)$, etc., are independent of each other so can be taken in any order. Thus, for example,

$$\vec{\nabla}\frac{\partial}{\partial t}f(\vec{r},t) = \frac{\partial}{\partial t}\vec{\nabla}f(\vec{r},t)\,,$$

where f is any function.

S-4 *(from PS-Problem 1)*

$(\vec{k}\cdot\vec{r})$ is written in the problem as $(2.09\,\mathrm{m}^{-1}x)$. The obvious conclusion is that $k_y = k_z = 0$ and hence that $k = \sqrt{k_x^2 + k_y^2 + k_z^2} = k_x = 2.09\,\mathrm{m}^{-1}$.

MODEL EXAM

$$\vec{\nabla}f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial g}{\partial y} + \hat{z}\frac{\partial h}{\partial z}$$

$$\vec{\nabla}\cdot\vec{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

$$\vec{\nabla}\times\vec{g} = \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\hat{z} + \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\hat{x} + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\hat{y}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{g}) = -\nabla^2\vec{g} + \vec{\nabla}(\vec{\nabla}\cdot\vec{g})\,;\ \nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi k_e\rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = 4\pi k_m j + c^{-2}\frac{\partial\vec{E}}{\partial t}$$

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2. Given these electric and magnetic fields:

$$\begin{array}{lll} E_x = E_0\cos\dfrac{2\pi}{\lambda}(y+ct), & E_y = 0, & E_z = 0, \\ B_x = 0, & B_y = 0, & B_z = 0. \end{array}$$

   a. Determine whether or not these fields satisfy the wave equations.

   b. Determine whether or not these fields satisfy Maxwell's Equations.

   c. If your answer to (a) and (b) is yes, what relationship must exist between the E and B amplitudes?

3. Given these electric and magnetic fields:

$$\begin{array}{lll} E_x = 0, & E_y = E_0\sin[2\pi\nu\left(\dfrac{x}{c} - t\right)], & E_z = 0, \\ B_x = 0, & B_y = 0, & B_z = B_0\sin[2\pi\nu\left(\dfrac{x}{c} - t\right)]. \end{array}$$

   (a), (b), (c): Repeat Problem 1 using the above components.

4. With:

   $E_x = E_0 \cos(kz - \omega t)$, $E_y = E_0 \cos(kz - \omega t)$, $E_z = 0$,

   write down the space-time dependence of the components of the magnetic field that will result in an electromagnetic wave that satisfies both the wave equations and Maxwell's equations.

5. A plane-polarized monochromatic electromagnetic wave of frequency ν has the electric field polarized in the z-direction and the wave propagates in the negative y-direction. Determine the components of $\vec{E}$ and $\vec{B}$ that satisfy Maxwell's equations and the wave equations.

Brief Answers:

1. See this module's *text*.
2. See Problem 4 in this module's *Problem Supplement*.
3. See Problem 5 in this module's *Problem Supplement*.
4. See Problem 6 in this module's *Problem Supplement*.
5. See Problem 7 in this module's *Problem Supplement*.