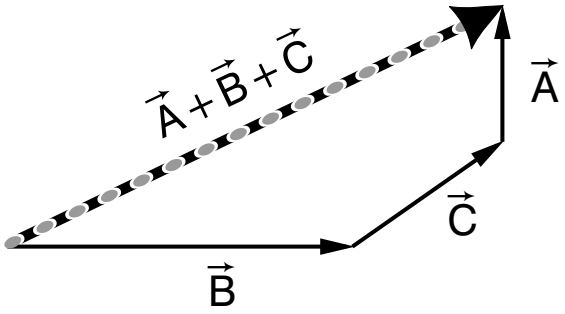
PHYSNET

MISN-0-2

SUMS, DIFFERENCES AND PRODUCTS OF VECTORS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SUMS, DIFFERENCES AND PRODUCTS OF VECTORS

by

J. S. Kovacs, Michigan State University

Title: **Sums, Differences and Products of Vectors**

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Version: 6/14/2002 Evaluation: Stage 0

Length: 1 hr; 32 pages

Input Skills:

1. Given some combination of angle(s) and leg length(s) for a triangle, determine the other angle(s) and leg length(s) (MISN-0-401).
2. Define the multiplication of a vector by a number and illustrate with a drawing (MISN-0-405).
3. Define the addition and subtraction of two vectors and illustrate with drawings (MISN-0-405).

Output Skills (Knowledge):

K1. In two dimensions, add any number of given vectors graphically.

K2. Given a vector's components in two dimensions, determine the magnitude of the vector and the angle it makes with each of the two coordinate axes, and conversely.

K3. In three dimensions, add and subtract given vectors expressed as (a) a triad of components and (b) in terms of unit vectors along the coordinate axes.

K4. Evaluate the scalar product of two given vectors, both in terms of the vectors' components along a fixed set of axes and in terms of the vectors' magnitudes and the angle between them.

K5. Evaluate the magnitude of the vector product of two given vectors in terms of: their magnitudes and the angle between them. Determine the direction by either the right-hand rule or the screw rule.

K6. Determine the vector product of two vectors, using their given components along a set of Cartesian coordinate axes.

External Resources (Optional):

1. Ronald C. Davidson and Jerry B. Marion, *Mathematical Preparation for General Physics and Calculus*, W. B. Saunders Co. (1973), (pp. 144-151).

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1. Introduction

1a. Scalars, Vectors, and Others. Most physical quantities can be classified mathematically as either scalars or vectors. Values of *scalar* quantities are single numbers, sizes, and this includes such quantities as temperature, work, distance, speed, musical pitch, and electric charge. Values of *vector* quantities have both size and direction, and this includes such quantities as displacement, velocity, acceleration, force, electric field, and momentum. There are other quantities whose values cannot be represented by either scalars or vectors; examples are color, stress, and inertia.

The rules for adding, subtracting, and multiplying scalars are just the familiar rules of arithmetic and algebra. In this module we deal with the rules for adding, subtracting, and multiplying vectors, rules that are used throughout professional science and engineering.

1b. Indicating Vectors Visually. Just as scalar quantities are referred to in graphs, drawings, text, and equations by alphabetic symbols, vector quantities are indicated by the use of arrows.

As a scalar example, suppose the length of a panel of concrete in a highway increases as its temperature rises. While describing the situation mathematically, we represent the length by the symbol L and the temperature by the symbol T. To indicate that the length is a function of temperature, we write $L(T)$.

As a vector example, suppose a particular force is changing with time. We write $\vec{F}$ to indicate that we are treating the force as a vector and t to show that we are treating time as a scalar. To indicate that the force is a function of time we write $\vec{F}(t)$. If we are only interested in the size of the force, not its direction, we write: $F(t)$.

When we want to indicate the size and direction of a vector quantity on a graph or a sketch of some physical phenomenon, we draw it as an arrow, a line with an arrowhead on one end (see Fig. 1). The length of the arrow shows the vector quantity's *magnitude* (its size) and the arrowhead shows its direction.

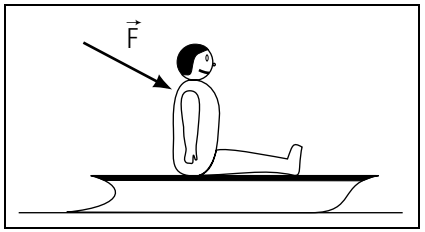


Figure 1. Force indicated on a diagram.

▷ In Fig. 1 the force scale is 10 lb per inch. Determine the magnitude of the force. *Help: [S-1]*[1]

2. Addition, Subtraction of Vectors

2a. General Features. When the rules of vector addition are properly used to add two vectors, the result is a third vector which usually has a different magnitude and direction than those of the two vectors being added. We here present two methods for adding two vectors: both produce the same result. In the *graphical* method you draw the vectors to scale and then construct the sum on the drawing and measure its magnitude and direction. The result only has the precision of your drawing and does not produce symbolic answers. In the *trigonometric* method you sketch the vectors and apply simple rules of trigonometry.

2b. Addition: Graphical Method. Here are the rules for adding two vectors graphically:

1. Take either of the vectors to be added and draw it on graph paper to scale, so it is an arrow of appropriate magnitude and direction.
2. Draw the arrow representing the second vector on the same graph, to the same scale, with the correct magnitude and direction, but with its tail at the head of the first arrow.
3. Draw a third arrow on the graph, going from the tail of the first arrow to the head of the second arrow. This third arrow is the sum of the other two: it is called the "resultant," which is the vector word for "sum."

These three steps are illustrated in Fig. 2, where each of the vectors being added is taken in turn as the first vector. Note that the resultant is the

[1]This notation indicates that help is available in sequence [S-1] of this module's *Special Assistance Supplement*.

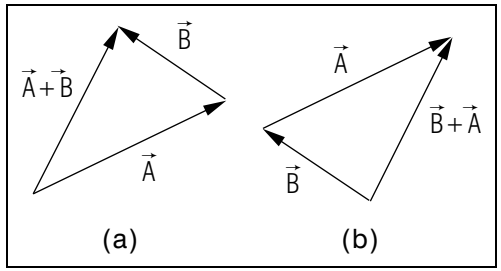


Figure 2. Two vectors are added two ways: (a) $\vec{A}$ is drawn first and then $\vec{B}$ is added to it; (b) $\vec{B}$ is drawn first and then $\vec{A}$ is added to it. The resultants in the two methods are the same: they have identical magnitudes and directions.

same in both cases.[2]

▷ Show, graphically, that the result of $\vec{A}+\vec{B}$ is the same as the result of $\vec{B}+\vec{A}$. *Help: [S-2]*

2c. Addition: Trigonometric Method. The trigonometric method of adding two vectors makes use of the law of cosines. This is illustrated in Fig. 3, where $\vec{R}$ is the "resultant" from adding vectors $\vec{A}$ and $\vec{B}$. The magnitude of $\vec{R}$ is determined from the right triangle whose sides are $A + B\cos\phi$ and $B\sin\phi$ and whose hypotenuse is R. Here ϕ is the angle B makes with A. Then:

$$R^2 = (A + B\cos\phi)^2 + B^2\sin^2\phi\,,$$

and the magnitude of R is:

$$R = \left(A^2 + B^2 + 2AB\cos\phi\right)^{1/2}\,. \qquad (1)$$

Most scientists and engineers either remember this equation or become proficient at quickly deriving it.[3] The angle that vector $\vec{R}$ makes with the vector $\vec{A}$ can also be expressed in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ by (see Fig. 3):

$$\theta = \tan^{-1}\left[\frac{B\sin\phi}{A + B\cos\phi}\right]. \qquad (2)$$

▷ Check Eqs. (1) and (2) by measuring and then calculating with the relevant quantities in Fig. 3. *Help: [S-3]*

[2]Mathematically, one says that vector addition is commutative.
[3]The derivation is in Sect. 4d of this module.

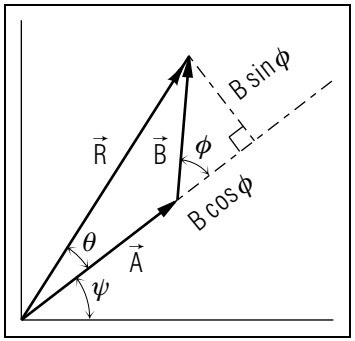


Figure 3. Using trigonometry to add two vectors.

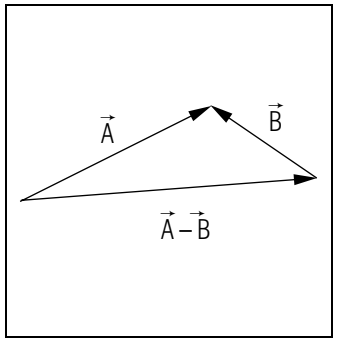


Figure 4. Illustration of vector subtraction.

2d. Subtraction. Subtraction is performed by using the rules for vector addition. That is, the subtraction of one vector from another is accomplished by adding the negative of the vector being "subtracted." The negative of a vector is most easily seen from its geometrical representation where the head and the tail of the arrow are interchanged. The magnitude of the vector is unchanged but its direction reverses. The subtraction $\vec{A}-\vec{B}$ is illustrated in Fig. 4.

Note also in Fig. 3 the difference $\vec{R}-\vec{B}$ must be $\vec{A}$ since $\vec{A}+\vec{B}=\vec{R}$. If the head and tail of $\vec{B}$ are interchanged, the result is $-\vec{B}$ and that vector can then be added to $\vec{R}$: $\vec{R}+(-\vec{B})=\vec{A}$.

▷ Try it geometrically and see if you get $\vec{A}$!

3. The Product of Two Vectors

3a. Two Kinds of Products. There are two kinds of multiplication possible for two vectors: one produces a scalar result while the other produces a vector result. Important physical quantities are usually obtained by such multiplication. For example, work is often obtained as a *scalar* product of force and displacement vectors, while angular momentum is often obtained as a *vector* product of radius and momentum vectors. Each of these kinds of multiplication has its own rules.

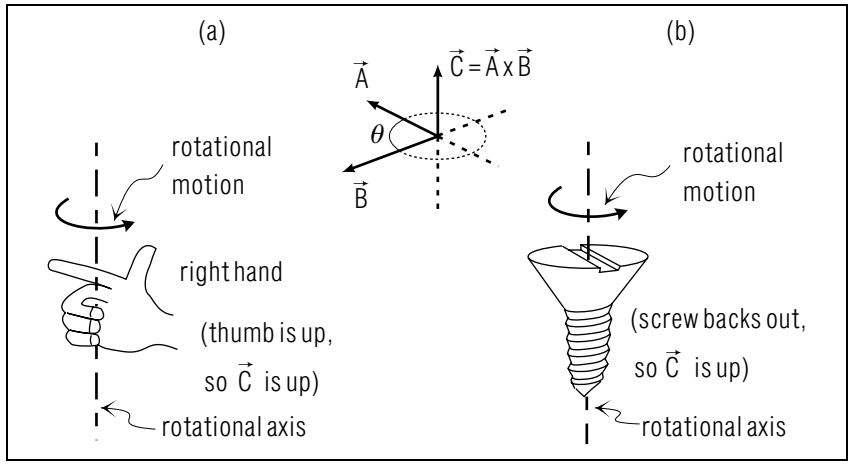


Figure 5. Two rules for finding the direction of $\vec{C} = \vec{A} \times \vec{B}$ by: (a) the "right hand" rule; (b) the "screw" rule.

3b. The Scalar Product. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is the magnitude of $\vec{A}$ times the magnitude of $\vec{B}$ times the cosine of angle between them (this angle is usually denoted θ). The scalar product is sometimes called the "dot product" because of the way it is indicated visually:

$$\text{scalar product } \equiv \vec{A} \cdot \vec{B} = AB\cos\theta. \tag{3}$$

▷ Show that the scalar product of two vectors is zero if the two are perpendicular.

▷ Under what other circumstances will the scalar product be zero? *Help: [S-4]*

▷ Let $\vec{R} = \vec{A} + \vec{B}$ and let θ be the angle between $\vec{A}$ and $\vec{B}$. Use $R = \sqrt{R^2} = \sqrt{\vec{R} \cdot \vec{R}}$ to show that $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$. *Help: [S-5]* Note: Some professionals use this method to produce the "adjacent sides and included angle" trig formula.

3c. The Vector Product. The vector product of, say, $\vec{A}$ and $\vec{B}$, is itself a vector and we will show separately how to find its magnitude and direction. The vector product is sometimes called the "cross product" because of the way it is indicated visually:

$$\vec{C} = \vec{A} \times \vec{B}.$$

The magnitude of $\vec{C}$ is defined as magnitude of $\vec{A}$ times the magnitude of $\vec{B}$ times the sine of the angle between them, as measured counterclockwise from the direction of $\vec{A}$ to the direction of $\vec{B}$:

$$|\vec{C}| = |\vec{A} \times \vec{B}| = AB\sin\theta. \tag{4}$$

The direction of the vector $\vec{C}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ and that narrows it to one of two directions. We will here describe two rules for obtaining the exact direction.

The "right-hand rule":

1. Extend the index finger of your right hand in the direction of the first vector ($\vec{A}$ in the example) with the rest of the fingers closed.
2. Rotate the index finger and hand until the index finger aligns with the second vector ($\vec{B}$ in the above example). That is, the first of the two vectors denoted in the cross product is rotated toward the second through the smaller angle between them ($\vec{A}$ rotated toward $\vec{B}$ such that θ in Fig. 5 decreases) and the curled fingers of the right hand follow this rotation.
3. The direction of the extended thumb gives you the direction of $\vec{C}$ (see Fig. 5).

The "screw" rule:

1. Imagine placing vectors $\vec{A}$ and $\vec{B}$ tail-to-tail, as in Fig. 5.
2. Imagine a screw with right hand threads placed where the tails of the two vectors come together, with the axis of the screw perpendicular to the plane formed by the two vectors (see Fig. 5).
3. Imagine a screwdriver placed in the screw with the axis of the screwdriver being along the axis of the screw.
4. Imagine a spot on the screw, next to vector $\vec{A}$.
5. Turn the screwdriver through the shortest angle so the spot is now next to vector $\vec{B}$. The direction the screw went (in or out) is the direction of the product vector $\vec{C}$ (see Fig. 5).

▷ Show that the vector product of two vectors is zero if the two are parallel.

▷ Show that the vector product of a vector with itself is zero.

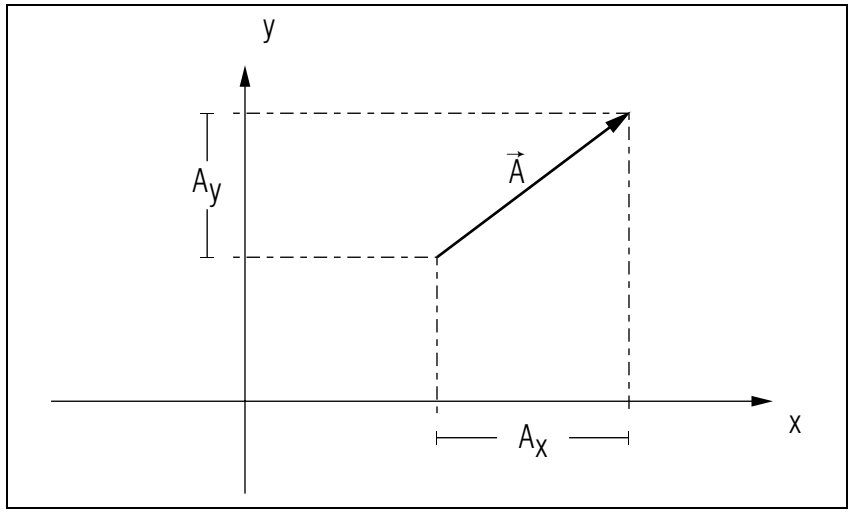


Figure 6. Cartesian components of a vector.

4. Using Cartesian Components

4a. The Components. You can use the Cartesian components of vectors to add and subtract the vectors and to take their scalar and vector products. Here is the process for obtaining the components of vectors so the components can be combined according to the rules for addition, subtraction, and scalar and vector multiplication. For two vectors, say $\vec{A}$ and $\vec{B}$ (refer to Fig. 6 in each step):

1. Draw a set of Cartesian (mutually perpendicular) coordinate axes and label them x, y, and z, as usual. Choose the origin and orientation of the axes for convenience in the problem being tackled (this will become automatic with experience). The results will be independent of your choice of axes.

2. Draw lines from each end of both $\vec{A}$ and $\vec{B}$ to the axes, such that these lines are perpendicular to the axes. Thus for the two vectors $\vec{A}$ and $\vec{B}$ there will be four such lines to each of the three axes (12 lines in all).

3. Label A_x the distance along the x-axis between the two perpendiculars from the ends of $\vec{A}$. Label A_y and A_z the similar distances along the y- and z-axes for $\vec{A}$. Do the same for $\vec{B}$, labeling the projections along the axes B_x, B_y, and B_z.

4. Call A_x, A_y, and A_z the *components* of $\vec{A}$ and B_x, B_y, and B_z the *components* of $\vec{B}$.

The components of a vector are frequently presented as a triad of numbers, in the specific order: x-component, y-component, z-component. Thus for $\vec{A}$ we write:

$$\vec{A} = (A_x,\, A_y,\, A_z). \tag{5}$$

▷ Find the numerical values of the components of the vector in Fig. 6. *Help: [S-6]*

4b. Addition. If the two vectors to be added are $\vec{A}$ and $\vec{B}$, with components $(A_x,\, A_y,\, A_z)$ and $(B_x,\, B_y,\, B_z)$, then the resultant vector $\vec{R}$ has components $(A_x + B_x, A_y + B_y, A_z + B_z)$. That is,

$$\begin{aligned} R_x &= A_x + B_x\,, \\ R_y &= A_y + B_y\,, \\ R_z &= A_z + B_z\,. \end{aligned} \tag{6}$$

Thus if the Cartesian components of two vectors are known, the Cartesian components of the resultant are easy to compute. The verification of this equation is left to an Appendix.

▷ Vector $\vec{A} = (1, 0, 1)$ while $\vec{B} = (0, 1, -1)$. Determine the components of $R = \vec{A} + \vec{B}$. *Help: [S-7]*

4c. Subtraction. If $\vec{B}$ is to be subtracted from $\vec{A}$ to produce $\vec{C}$,

$$\vec{C} = \vec{A} - \vec{B}$$

we just use the rules of addition but with the sign of the components of $\vec{B}$ made negative:

$$\begin{aligned} C_x &= A_x - B_x\,, \\ C_y &= A_y - B_y\,, \\ C_z &= A_z - B_z\,. \end{aligned} \tag{7}$$

▷ Example: Vector $\vec{A} = (1, 0, 1)$ while $\vec{B} = (0, 1, -1)$. Determine the components of $R = \vec{A} - \vec{B}$. *Help: [S-8]*

4d. Scalar Product. In terms of Cartesian components, the scalar or "dot" product of A and B to form C is:

$$C = \vec{A} \cdot \vec{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z. \tag{8}$$

▷ Vector $\vec{A} = (1, 0, 1)$ while $\vec{B} = (0, 1, -1)$. Use Cartesian components to determine: $R = \vec{A} \cdot \vec{B}$. *Help: [S-9]*

4e. Vector Product. The six remaining products of the components of two vectors may be combined so that they form the three components of another vector which is used in almost all areas of physics and technology. This new vector is called the *vector product* of the two combining vectors and is written with a multiplication sign. Here is how the vector $\vec{C}$ is written as the vector product of $\vec{A}$ and $\vec{B}$:

$$\vec{C} = \vec{A} \times \vec{B}\,. \tag{9}$$

In terms of Cartesian components:

$$\begin{aligned} C_x &= A_yB_z - A_zB_y \\ C_y &= A_zB_x - A_xB_z \\ C_z &= A_xB_y - A_yB_x \end{aligned} \tag{10}$$

The mnemonic for remembering the order of the subscripts on these components is to note that, starting from left to right, the first three subscripts are always cyclic permutations of xyz (xyz, yzx, zxy).[4]

▷ Vector $\vec{A} = (1, 0, 1)$ while $\vec{B} = (0, 1, -1)$. Determine the components of $\vec{R} = \vec{A} \times \vec{B}$. *Help: [S-10]*

5. Using Unit Vectors

5a. Cartesian Unit Vectors. A vector can be written as the sum of its components through the use of "unit vectors," which are three vectors of unit length that each point outward along one of the three Cartesian coordinate axes you are using. We write these unit vectors as $\hat{x}$, $\hat{y}$ and $\hat{z}$, where $\hat{x}$ is a vector of unit length that points in the direction

[4]Note that the combination $A_yB_z - A_zB_y$ is the x-component of this vector, expressed with respect to the same coordinate system. Two other pairs of products complete the groupings (and, incidentally, together with the combinations that form the scalar product, exhaust the possibilities of pairs of products of components of A and $\vec{B}$.

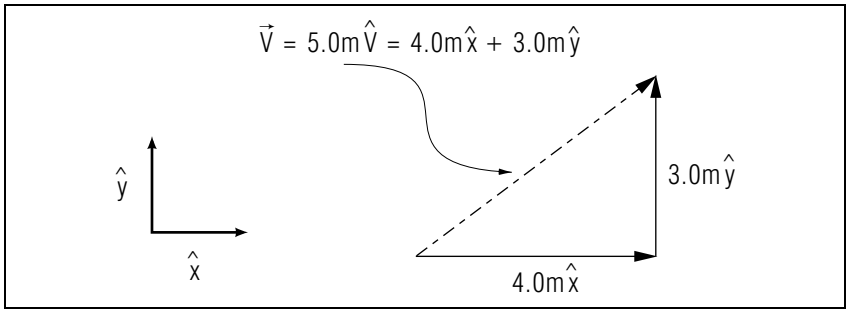


Figure 7. Illustration of combining unit vectors and a vector's components.

of increasing values along the x-axis, $\hat{y}$ similarly along the y-axis, and $\hat{z}$ along the z-axis.[5] Since the unit vectors are of unit length, a vector $\vec{A}$ with components (A_x, A_y, A_z) can be written as the sum of three vectors:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}. \tag{11}$$

This addition is illustrated for two dimensions in Fig. 7.

5b. Products Using Cartesian Unit Vectors and Components. ▷ Use the definitions of products of vectors, Eqs. (8) and (10), to show that:

$$\begin{aligned} &\hat{x}\cdot\hat{x} = 1, \quad &&\hat{y}\cdot\hat{y} = 1, \quad &&\hat{z}\cdot\hat{z} = 1, \\ &\hat{x}\cdot\hat{y} = 0, \quad &&\hat{x}\cdot\hat{z} = 0, \quad &&\hat{y}\cdot\hat{z} = 0, \\ &\hat{x}\times\hat{y} = \hat{z}, \quad &&\hat{y}\times\hat{z} = \hat{x}, \quad &&\hat{z}\times\hat{x} = \hat{y}, \\ &\hat{x}\times\hat{z} = -\hat{y}, \quad &&\hat{x}\times\hat{x} = 0, \text{ etc.} \end{aligned} \tag{12}$$

Note that, in the vector product, the order (xyz), in any cyclic permutation, has a plus sign on the right hand side. Any other order has a minus sign.[6] *Help: [S-11]*

5c. The Length of a Vector in Terms of its Components. To find the length of a vector $\vec{A}$ write:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

[5]Different authors use different notations for unit vectors along the axes of a Cartesian coordinate system. We use $\hat{x}$, $\hat{y}$ and $\hat{z}$ for unit vectors along the x-, y-, and z-axes, which is a commonly used notation. Other authors have used such notations as: ($\hat{i}$, $\hat{j}$, $\hat{k}$), (**i**, **j**, **k**), ($\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$), and ($\hat{u}_x$, $\hat{u}_y$, $\hat{u}_z$). All such sets are equivalent.

[6]This provides a test for whether a Cartesian coordinate system is *right-handed*, which is what we use throughout science and engineering. An improperly defined, *left-handed*, system would produce $\hat{x} \times \hat{y} = -\hat{z}$.

and take the scalar product of $\vec{A}$ with itself:

$$\begin{aligned}\vec{A}\cdot\vec{A} &= (A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\cdot(A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\\ &= A_x^2+A_y^2+A_z^2\,.\end{aligned}$$

where we have used the products of unit vectors, Eq. (12).

Now from the definition of the scalar product, Eq. (8), we know that $\vec{A}\cdot\vec{A}=A^2$, which is the square of the length of vector $\vec{A}$. Then the length of A is given in terms of its components by the square root of the product of A with itself, $\vec{A}\cdot\vec{A}$:

$$A=\sqrt{A_x^2+A_y^2+A_z^2}\,. \tag{13}$$

▷ Show that: $|\vec{A}\times\vec{B}|=|\vec{B}\times\vec{A}|$. *Help: [S-17]*

▷ Show that: $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$. *Help: [S-18]*

5d. Determinant Form of the Vector Product. Unit vectors can be used to write the vector product in determinant form. Some people find easier to remember these forms:

$$\vec{C}=\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ A_x&A_y&A_z\\ B_x&B_y&B_z\end{vmatrix} \tag{14}$$

The minors of the unit vectors as co-factors are the appropriate components. *Help: [S-16]*

▷ Use the determinant method to show that $\hat{x}\times\hat{y}=\hat{z}$. *Help: [S-12]*

▷ Given $\vec{A}=5\hat{x}-2\hat{y}$ and $\vec{B}=\hat{x}+\hat{y}+3\hat{z}$), show that $\vec{A}\times\vec{B}=-6\hat{x}-15\hat{y}+7\hat{z}$. *Help: [S-13]*

5e. Representing a Vector by Magnitude and Direction. It is often convenient to to write a vector as the product of a single magnitude and a single direction rather than as the sum of its Cartesian Components. First we define the magnitude of a vector:

$$A\equiv|\vec{A}|=\sqrt{A_x^2+A_y^2+A_z^2}\,. \tag{15}$$

Note that the magnitude it always positive. Next we define a unit vector that is in the direction of the vector we are representing:

$$\hat{A}=\equiv\frac{\vec{A}}{|\vec{A}|}\,. \tag{16}$$

Note that $\hat{A}$ has unit length because we have divided $\vec{A}$ by its own length. Then we can rewrite Eq. (16) to show the vector as the product of a magnitude and a direction, which is entirely equivalent to the representation of the vector in Cartesian components:

$$\vec{A}=A\hat{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\,. \tag{17}$$

Finally, we write the magnitude and direction of the vector product $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ in terms of the magnitudes and directions of $\vec{A}$ and $\vec{B}$. First, the magnitude is:

$$\begin{aligned}C&=\left[(A_yB_z-A_zB_y)^2+(A_zB_x-A_xB_z)^2+(A_xB_y-A_yB_x)^2\right]^{1/2}\\ &=AB\sin\theta\,,\end{aligned} \tag{18}$$

where θ is the counter-clockwise angle between the directions of $\vec{A}$ and $\vec{B}$. The direction of the vector product is:

$$\hat{C}=\frac{\vec{C}}{C}=\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}=\frac{(A_yB_z-A_zB_y)\hat{x}+\ldots}{[(A_yB_z-A_zB_y)^2+\ldots]^{1/2}}\,. \tag{19}$$

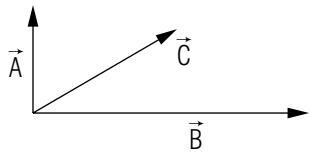
Acknowledgments

Professor James Linneman made several helpful suggestions. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

Note: problems 13 (part), 21, and 22 (part) also occur in this module's *Model Exam.*

1.



Three vectors, $\vec{A}$, $\vec{B}$, and $\vec{C}$, all in a plane, are shown sketched above. By arranging the vectors in head-to-tail fashion, sketch:

a. $\vec{A}+\vec{B}$, to which then add $\vec{C}$;
b. $\vec{A}+\vec{C}$, to which then add $\vec{B}$;
c. $\vec{B}+\vec{C}$, to which then add $\vec{A}$;
d. $\vec{A}-\vec{B}$.

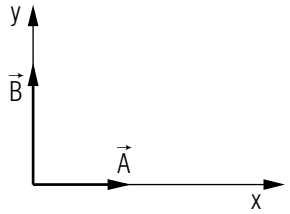
2. Write down the ordered triad of numbers which represents a vector of length 6 units and that lies along the y-axis of a fixed coordinate system.

3. Write down the ordered triad of numbers which represents a vector of length 6 units that lies in the xy-plane, making an angle of 30° with the x-axis.

4. Express the vectors in problems 2 and 3 above in terms of the fixed unit vectors along the Cartesian axes.

5. A particular vector is given as a zero or null vector. Write down the triad that represents this vector.

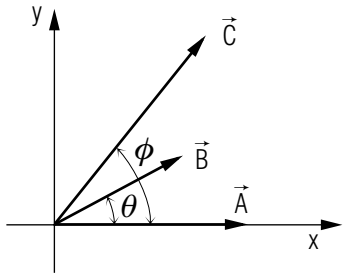
6.



$|\vec{A}| = 4$ units, $|\vec{B}| = 5$ units, directions as in the sketch.

a. Graphically add $\vec{A}$ and $\vec{B}$.
b. Compute the magnitude of the resultant.
c. Determine the direction of the resultant, expressed as:
   (i) The angle between the vector and the positive x-axis, measured counter-clockwise from the positive x-axis; this is the standard representation of a vector's direction.
   (ii) The angle between the vector and the positive y-axis, measured clockwise from the positive y-axis; this is a compass bearing if the positive y-axis represents due north and the positive x-axis represents due east.

7.



$|\vec{A}| = 4$ units, $|\vec{B}| = 3$ units, $\theta = 30°$, $|\vec{C}| = 5$ units, $\phi = 60°$ (see sketch).

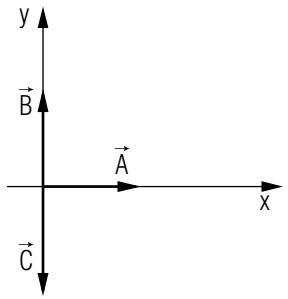
a. Express $\vec{A}$, $\vec{B}$, and $\vec{C}$ in component form. *Help: [S-15]*
b. Using these components, evaluate $\vec{A}+\vec{B}+\vec{C}$.
c. Similarly, evaluate $\vec{A}+\vec{B}-\vec{C}$.

8. A vector of unit length, $\hat{x}$, is parallel to the x-axis of a Cartesian coordinate system while another vector of unit length, $\hat{y}$, is along the y-axis. Evaluate the scalar products $\hat{x}\cdot\hat{x}$, $\hat{x}\cdot\hat{y}$, and $\hat{y}\cdot\hat{y}$.

9. Expressing two vectors in terms of unit vectors along the coordinate axes, $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$, show that the scalar product may be written as $A_xB_x + A_yB_y + A_zB_z$ (use results that you got in problem 8, above.)

10. Consider two vectors, $\vec{A} = 4\hat{x} + 3\hat{y}$ and $\vec{B} = 5\hat{x} - 12\hat{y}$. Find: (a) the length of $\vec{A}$ starting from the definition of the scalar product; (b) the length of $\vec{B}$; and (c) the angle between $\vec{A}$ and $\vec{B}$.

11.



Assuming magnitudes (see sketch) $|\vec{A}| = |\vec{B}| = |\vec{C}| = R$, evaluate the following in terms of R:

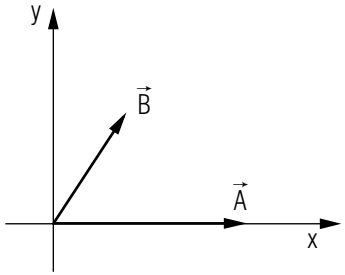
a. $(\vec{A} \times \vec{B}) \times \vec{C}$

b. $(\vec{B} \times \vec{A}) \times \vec{C}$

c. $\vec{A} \times (\vec{B} \times \vec{C})$

d. $\vec{C} \times (\vec{A} \times \vec{B})$

12.



Given the vectors $\vec{A}$ and $\vec{B}$:

a. Sketch the graphical representation of the vector sum of $\vec{A}$ and $\vec{B}$ using the head-to-tail addition method.

b. Using the geometrical definition of the scalar product, prove the Law of Cosines for the triangle formed by the above vector sum,

i.e.;

$$C^2 = A^2 + B^2 - 2\,A\,B\cos\gamma\,,$$

where C is the magnitude of $\vec{C}$, the sum $\vec{A}+\vec{B}$, γ is the angle opposite side C in the triangle ABC, and A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, respectively. *Help: [S-14]*

c. Use the definition of the magnitude of the vector product to prove the Law of Sines for the same triangle, i.e.;

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

where α, β and γ are the angles of the triangle that are opposite sides A, B and C.

13. A vector $\vec{R}$ has its tail at the origin and its head at the point $(2, 5, -3)$. Another vector $\vec{S}$ has its tail also at the origin, while its head is at the point $(2, -5, 0)$.

a. How long is the y-component of $\vec{R}$?

b. How long is the y-component of $\vec{S}$?

c. What are the coordinates of the head of the vector (-$\vec{R}$)?

d. What are the coordinates of the head of the vector (or, stated more simply, the coordinates of the vector) $\vec{R} + \vec{S}$?

e. What are the coordinates of the vector $\vec{R}$ - $\vec{S}$?

f. What is the length of the vector $\vec{R} + \vec{S}$?

g. What is the length of the vector $\vec{S}$ - $\vec{R}$?

h. What is the length of the vector $\vec{R}$?

14. Express vector $\vec{R}$ (of problem 13 above) as a single sum of components along the coordinate axes in terms of the unit vectors along those axes.[7]

15. Similarly express the vector $3\vec{S}$.

16. Do the same for the vector $3\vec{R} - 2\vec{S}$.

17. What angle does the vector $\vec{R} + \vec{S}$ make with the z-axis?

18. What angle does the vector $\vec{R} - \vec{S}$ make with the x-axis?

19. What is the cosine of the angle between the vector $\vec{R}$ and the y-axis?

20. Vector $\vec{R}$ has its tail at the origin of the coordinate system, while its head is at the point (2, 5, −3). Vector $\vec{S}$ likewise has its tail at the origin of the coordinate system, while its head is at point (2, −5, 0).

[7] Authors use various notations for Cartesian unit vectors. We use $\hat{x}$, $\hat{y}$ and $\hat{z}$ for unit vectors along the x-, y-, and z-axes, which is a commonly used notation. Some authors have used: $\hat{i}$, $\hat{j}$, $\hat{k}$; $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$; and $\hat{u}_x$, $\hat{u}_y$, $\hat{u}_z$. All such sets are equivalent.

a. What is the value of $\vec{R} \cdot \vec{S}$?

b. Using the result of (a), find the angle between the vectors $\vec{R}$ and $\vec{S}$.

c. Evaluate the magnitude of the vector $\vec{R} \times \vec{S}$, using the result of (b).

21. Vector $\vec{T}$ is 4 units long and directed along the x-axis. Vector $\vec{V}$ is 6 units long and is directed along the z-axis.

a. What is the direction of the vector $\vec{T} \times \vec{V}$?

b. What is the magnitude of the vector $\vec{T} \times \vec{V}$?

c. Express $\vec{T} \times \vec{V}$ in terms of the unit vectors along the coordinate axes.

22. Evaluate:

a. $\hat{x} \times \hat{z}$

b. $\hat{y} \cdot \hat{z}$

c. $\hat{z} \times \hat{z}$

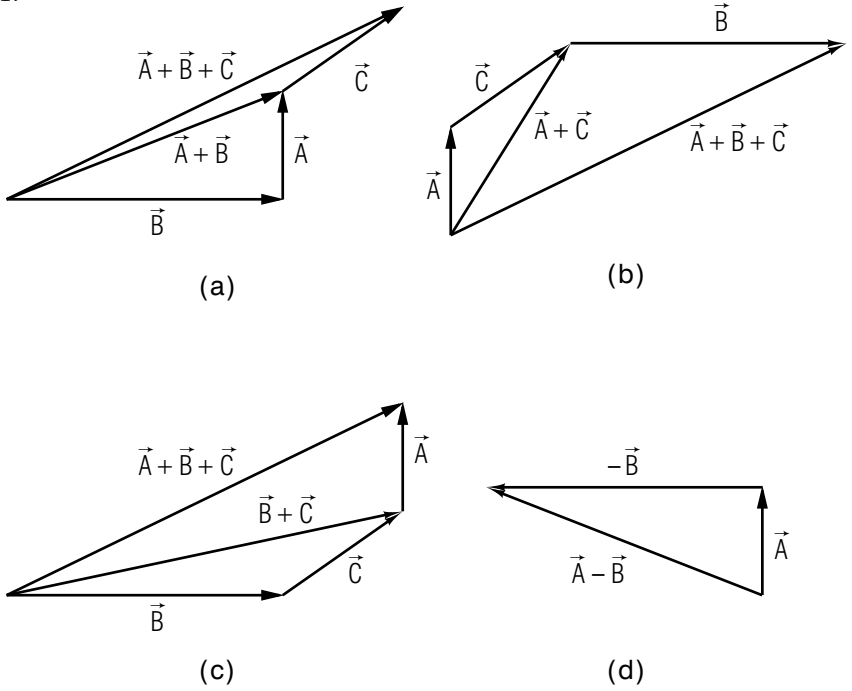
d. $\hat{y} \times \hat{z}$

e. $\hat{x} \cdot \hat{x}$

f. $\hat{x} \times \hat{y}$

Brief Answers:

1.



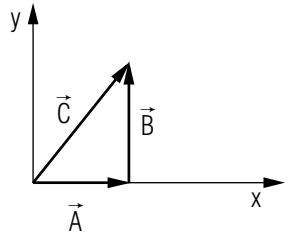
2. (0, 6, 0)

3. $(3\sqrt{3}, 3, 0)$

4. $\vec{A} = 6\hat{y}$, $\vec{B} = 3\sqrt{3}\hat{x} + 3\hat{y}$

5. (0, 0, 0)

6. a. $\vec{C}$ = resultant:



   b. $|\vec{C}| = 6.4$ units

   c. (i) 51° or 0.90 rad

      (ii) 39° or 0.67 rad

7. a. $\vec{A} = 4\hat{x}$, $\vec{B} = 2.60\hat{x} + 1.5\hat{y})$, $\vec{C} = 2.5\hat{x} + 4.3\hat{y})$

   b. $\vec{S} = \vec{A} + \vec{B} + \vec{C} = 9.1\hat{x} + 5.8\hat{y}$

   c. $\vec{D} = \vec{A} + \vec{B} - \vec{C} = 4.1\hat{x} - 2.8\hat{y}$

8. $\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{y} = 0$, $\hat{y} \cdot \hat{y} = 1$

10. $|\vec{A}| = 5$, $|\vec{B}| = 13$, $\theta = 104°$

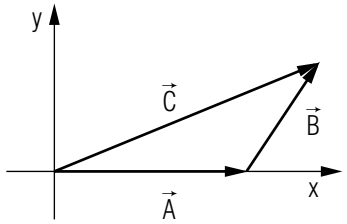
11. a. $(\vec{A} \times \vec{B}) \times \vec{C} = R^3\hat{x}$

   b. $(\vec{B} \times \vec{A}) \times \vec{C} = -R^3\hat{x}$

   c. $\vec{A} \times (\vec{B} \times \vec{C}) = 0$

   d. $\vec{C} \times (\vec{A} \times \vec{B}) = -R^3\hat{x}$

12. a.



13. a. 5 units

   b. 5 units

   c. $(-2, -5, 3)$

   d. $(4, 0 - 3)$

   e. $(0, 10, -3)$

   f. 5 units

   g. $\sqrt{109} = 10.44$ units

   h. $\sqrt{38} = 6.16$ units

14. $2\hat{x} + 5\hat{y} - 3\hat{z}$

15. $6\hat{x} - 15\hat{y}$

16. $2\hat{x} + 25\hat{y} - 9\hat{z}$

17. 127°

18. $\pi/2$

19. 0.811

20. a. -21 units

   b. 129.2°

   c. 25.71 units

21. a. along negative y-axis

   b. 24 units

   c. $-24\hat{y}$

22. a. $-\hat{y}$

   b. 0

   c. 0

   d. $\hat{x}$

   e. 1

   f. $\hat{z}$

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-1b)*

Using a ruler, we find the $\vec{F}$ vector to be 0.5 in long so $F = 10\,\text{lb/in} \times 0.5\,\text{in} = 5\,\text{lb}$.

S-2 *(from TX-2b)*

The easiest way is to put a sheet of paper over Fig. 2 and put both sheets together against a window pane that is illuminated mainly from the other side. Trace the first resultant onto the clean sheet, then slide it over to the other resultant and see if the magnitude and direction are identical (up to your ability to trace and compare visually).

S-3 *(from TX-2c)*

In Fig. 3, measure A, B, and R with a ruler and measure the angles θ and ϕ with a protractor. Then use your calculator to finish the job, making sure you have set it for the proper choice of radians or degrees.

S-4 *(from TX-3b)*

The magnitude of one or both is zero.

S-5 *(from TX-3b)*

$$R = \left[(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})\right]^{1/2} = \left(A^2+B^2+2\vec{A}\cdot\vec{B}\right)^{1/2}$$

S-6 *(from TX-4a)*

We find: $A_x = 1.00\,\text{inches}$, $A_y = 0.75\,\text{inches}$.

S-7 *(from TX-4b)*

$$(1,\,0,\,1)+(0,\,1,-1)=(1,\,1,\,0)$$

S-8 *(from TX-4c)*

$$(1,\,0,\,1)-(0,\,1,-1)=(1,-1,\,2)$$

S-9 *(from TX-4d)*

$$(1,\,0,\,1)\cdot(0,\,1,-1)=1\cdot 0+0\cdot 1+1\cdot(-1)=-1$$

S-10 *(from TX-4e)*

$$(1,\,0,\,1)\times(0,\,1,-1)=(-1,\,1,\,1)$$

S-11 *(from TX-5a)*

Use the "right hand" rule or the "screw" rule. You could also work with components: $\hat{x} = (1,0,0)$, $\hat{y} = (0,1,0)$, and $\hat{z} = (0,0,1)$. Then, for example, the x-component of $\hat{x}\times\hat{y}$ is: $(\hat{x}\times\hat{y})_x = (\hat{x})_y(\hat{y})_z - (\hat{x})_z(\hat{y})_y = 0\cdot 0 - 0\cdot 1 = 0$.

S-12 *(from TX-5b)*

$$\hat{x}\times\hat{y} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ (\hat{x})_x & (\hat{x})_y & (\hat{x})_z \\ (\hat{y})_x & (\hat{y})_y & (\hat{y})_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

S-13 *(from TX-5b)*

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 5 & -2 & 0 \\ 1 & 1 & 3 \end{vmatrix}$$

S-14 *(from PS, problem 12b)*

If you get

$$C^2 = A^2 + B^2 + 2AB\cos\gamma$$

then you did not pay attention to the definition of γ in the statement of the problem. The definition of γ is taken from the usual statement of the relationship between two sides and the included angle in a triangle.

If you do not know how to obtain $\cos(\pi - \gamma)$ from $\cos(\gamma)$, see "trigonometric functions" in this volume's *Index*.

S-15 *(from PS, problem 7a)*

Look at the figure in the problem statement. To find C_x, drop a vertical line from the tip of $\vec{C}$ to the x-axis, so the line is perpendicular to the x-axis. Label this line C_y and label the x-axis from the origin out to this line C_x. Notice the right-angle triangle formed by the three lines C, C_x, and C_y (the vertical line down from the tip of $\vec{C}$). You know the values of C and ϕ so simply apply trigonometry to find C_x. If you don't remember how, see "trigonometric functions" in this volume's *Index*.

S-17 *(from TX-5c)*

You can show it any of three ways. We suggest that you do all three and then choose the method that suits you best. The ways are: (1) by expanding $|\vec{A} \times \vec{B}|$ and $|\vec{B} \times \vec{A}|$ in Cartesian components and showing that the two are equal (the absolute magnitude of a vector is given in Eq. (13)); or (2) by showing that the answer is unchanged if you interchange $\vec{A}$ and $\vec{B}$ in Eq. (4); or (3) by taking the absolute magnitude of each side of $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, in which case the absolute magnitude of the right side deletes the over-all minus sign and the proof is complete.

S-18 *(from TX-5c)*

You can show it either of two ways. We suggest that you do both and then choose the method that suits you best. The ways are: (1) by expanding the cross-products in Cartesian components; or (2) by using the fact that $\vec{A} \times \vec{B}$ has the same magnitude as $\vec{B} \times \vec{A}$ (see *S-17*)), in which case the two can differ only by direction, and then using the "tight-hand rule" or the "screw" rule to show that the direction of either one is opposite to that of the other.

MODEL EXAM

1. See Output Skills K1-K6 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.

2. A vector $\vec{R}$ has its tail at the origin and its head at the point $(2, 5, -3)$. Another vector $\vec{S}$ has its tail also at the origin, while its head is at the point $(2, -5, 0)$.

   a. What is the length of the vector $\vec{R} + \vec{S}$?

   b. What is the length of the vector $\vec{S} - \vec{R}$?

   c. What is the length of the vector $\vec{R}$?

3. Vector $\vec{T}$ is 4 units long and directed along the x-axis. Vector $\vec{V}$ is 6 units long and is directed along the z-axis.

   a. What is the direction of the vector $\vec{T} \times \vec{V}$?

   b. What is the magnitude of the vector $\vec{T} \times \vec{V}$?

   c. Express $\vec{T} \times \vec{V}$ in terms of the unit vectors along the coordinate axes.

4. Evaluate:

   a. $\hat{x} \times \hat{z}$

   b. $\hat{y} \cdot \hat{z}$

   c. $\hat{z} \times \hat{z}$

   d. $\hat{y} \times \hat{z}$

   e. $\hat{x} \cdot \hat{x}$

   f. $\hat{x} \times \hat{y}$

Brief Answers:

1. See this module's *text*.

2-4. See problems 13, 21, and 22 in this module's *Problem Supplement*.