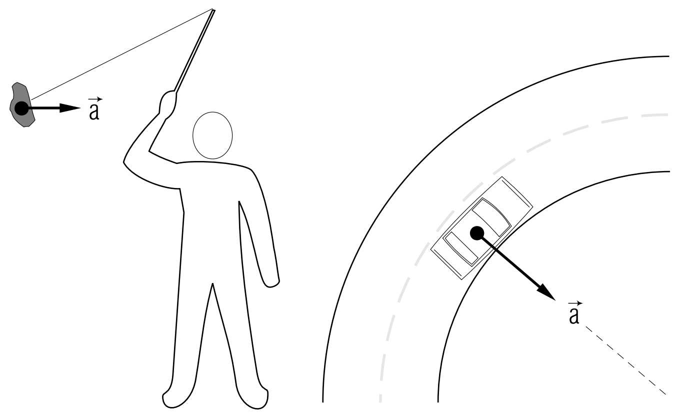
PHYSNET

MISN-0-17

ACCELERATION AND FORCE IN CIRCULAR MOTION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ACCELERATION AND FORCE IN CIRCULAR MOTION

by

Peter Signell

Title: **Acceleration and Force in Circular Motion**

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Input Skills:

1. Draw one-body force diagrams and solve problems involving Newton's second law (MISN-0-16).
2. Solve problems involving circular kinematics (MISN-0-9).
3. Determine the resultant force produced by a given set of forces (MISN-0-14).

Output Skills (Knowledge):

K1. Vocabulary: centripetal force, centripetal acceleration, centrifugal force, banking angle, ideal banking angle.

K2. Derive the expression for the ideal banking angle for uniform circular motion.

Output Skills (Rule Application):

R1. Produce non-gravitational accelerations as numbers times g (called "g's" or "gees").

Output Skills (Problem Solving):

S1. Draw one-body force diagrams for, and solve, problems involving forces, velocity, period, frequency, radius, and mass for an object in uniform circular motion.

S2. Apply the expression for the ideal banking angle to problems involving uniform circular motion.

Post-Options:

1. "Classical Mechanics in Rotating Frames of Reference: Effects on the Surface of the Earth" (MISN-0-18).

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New authors, reviewers and field testers are welcome.

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1. Introduction

Here are some questions of the types we can answer from a study of acceleration and force in uniform circular motion: Why are highway curves banked, and what characteristics of vehicles and terrain determine the design angle? What happens if a car or truck does not match the vehicle characteristics that were assumed by the highway department in designing a particular curve? Why do drivers of mopeds, bicycles, and motorcycles lean while going around a corner? How much should they lean? What happens if they don't?

In the film, *2001: A Space Odyssey*, the space station rotates so as to simulate the force of gravity as we know it at the earth's surface (see Fig. 4). The space station is in the shape of two large wheels connected by an axle. How does it simulate the gravitational force and at what rate must it turn? Is it necessary to vary the rate according to the weight of each space-person?

Suppose you tie a rock on the end of a long string and then whirl it around your head. What governs the angle of the string?

Airplane pilots talk about "g" forces. What are they, and why are they important to pilots? How are they measured and what are their (obviously important) physiological effects?

2. Acceleration and Force

2a. The Circle of Motion. When an object is traveling along a circular arc, we talk about its "circle of motion," whether the object is traveling completely around a circle or only around part of a circle. If the motion is circular only on an arc, we mentally extend the arc to make a complete circle, and that is the "circle of motion."

2b. Uniform Circular Motion. For any object in uniform circular motion, its acceleration is radially inward, pointing precisely toward the center of the circle of motion, as in the example in Fig. 1. The magnitude

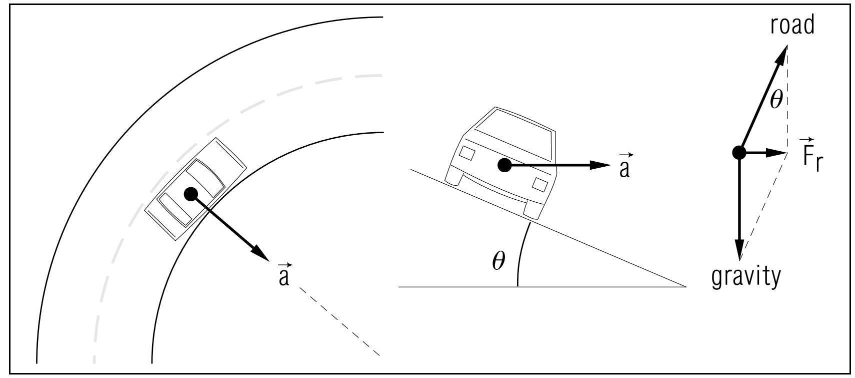


Figure 1. A car traverses a quarter-circle turn, traveling at constant speed so its acceleration $\vec{a}$ is purely radial. The car's speed is such that the road exerts only a normal force on the car. The turn is banked at angle θ, which is also the angle by which the road force is off the vertical. The resultant (net, total) force on the car is labeled $\vec{F}_r$.

of the object's acceleration is $a = v^2/r$, where v is the object's speed and r is its distance from the center of the circle of motion.[1] If you are given that an object has many forces on it, and that it is in uniform circular motion, then you know that the object's acceleration is toward the center of the circle. By Newton's second law, this means the *resultant* force on the object must also be toward the center of the circle of motion. This is illustrated in Fig. 1.

2c. Example 1: A Car on a Turn. Our first example of uniform circular motion is a car that is traveling at constant speed around a highway curve (see Fig. 1). Geometrically, the curve is a circular arc which we can mentally extend to make a complete imaginary circle. While the car is on the curve, it is maintaining a constant radius from the center of that imaginary circle. If we were to draw the car's path on an aerial photo, using a drafting compass, one leg of the compass would be at the center of that circle and the other on the car's path (see Fig. 1). Because the car is traveling at constant speed, the acceleration $\vec{a}$ is exactly toward the center of the circle of motion (see Fig. 1). Since $\vec{F}_r = m\vec{a}$, the resultant force on the car must also be toward the center of the circle of motion (see Fig. 1). We have drawn the force of the road on the car as being normal to the road: this will be the case if the car is traveling at just the right

[1] See "Kinematics: Circular Motion" (MISN-0-9).

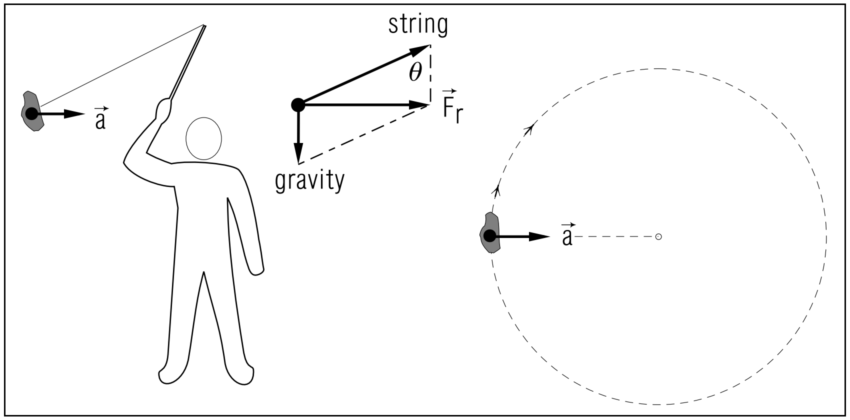


Figure 2. A string-restrained rock is whirled about the person's body axis at a constant speed. The force of the string on the rock is off the vertical by the angle θ.

speed (more on this later).

$\triangleright$ A 2000 lb car is traveling at 50 mph around the curve illustrated in Fig. 1, with $r = 390$ ft. Show that the acceleration of the car, traveling at a constant 50 mph, is $a = 0.43\,g$. That is, the acceleration is 43% of the usual acceleration of gravity for objects in free fall. *Help: [S-7]*

2d. Example 2: Rock on a String. Our second example is the case of a rock being whirled around the body at the end of a string as in Fig. 2. The string will only produce a force along its physical direction, as shown in the figure. We assume the rock is being made to travel at constant speed so its acceleration is $a = v^2/r$ and is exactly toward the center of the circle of motion. By $\vec{F}_r = m\vec{a}$, the resultant force on the rock must also be toward the center of the circle of motion (see Fig. 2).

2e. Example 3: A Bicyclist Rounding a Corner. Our third example is the case of a bicyclist tilting sideways while rounding a corner, as in Fig. 3. We could equally well have used a person rounding a corner while doing any of such diverse things as running, riding a moped or a motorcycle, or skiing. The point is that the person does not want whatever is supporting the body to exert a sideways force, one which would tend to throw the person off the support sideways. Thus in Fig. 3 we have assumed the person tilts to an angle such that the support force on the body has no sideways component. We also assume the forward speed is constant so the acceleration is exactly toward the center of the circle of motion. Then, by $\vec{F}_r = m\vec{a}$, the resultant force on the person must also be exactly toward the center of the circle of motion (see Fig. 3).

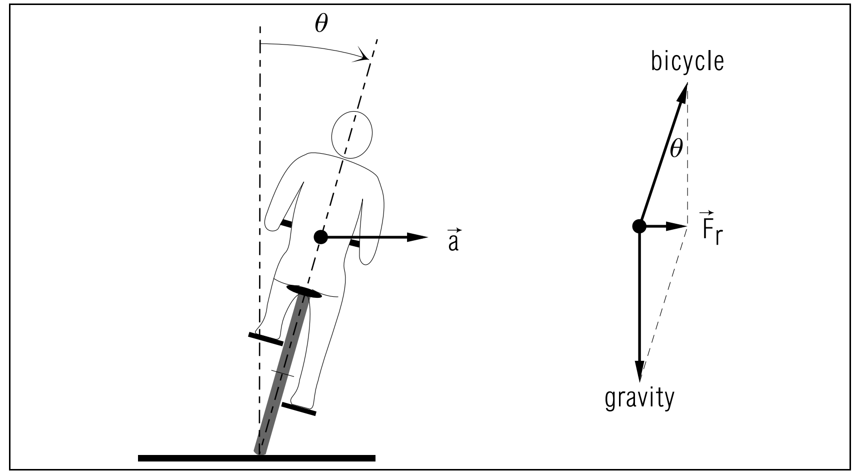


Figure 3. A bicyclist tilts sideways rounding a corner. The force of the bicycle on the bicyclist is off the vertical by the angle θ.

3. The Proper Highway Banking Angle

3a. A Sideways Force is Undesirable. If the roadway on a highway is properly banked, as in Fig. 1, the *roadway* will exert no sideways force on the car as long as the car maintains the proper speed. This is highly desirable since a sideways force can cause the car to start to accelerate (slide) sideways. This will happen if the sideways force exceeds the maximum sustainable force of non-sliding friction. Of course the maximum sustainable non-sliding frictional force depends drastically on the condition of the tires and the surface of the road. The maximum sustainable frictional force can be close to zero in a snowstorm or ice storm, or even in a sudden deluge from a summer rainstorm.

3b. Deducing the Angle. Our condition for the proper banking angle of a highway turn is that the road should only exert a normal force on the car, as in Fig. 1. Our problem is to deduce that proper banking angle, indicated by the symbol θ in Fig. 1. Data available for the calculation include the radius of the turn and the mass and speed of the vehicle.

Here are the five steps:

1. Combine the radius and speed to get the magnitude of the car's acceleration. The direction of the acceleration is known so now we know $\vec{a}$.
2. Use $\vec{F}_r = m\vec{a}$ to get $\vec{F}_r$.
3. Require that $\vec{F}_r$ be the result of adding the known gravity force to a *normal* road force, as shown in the one-body diagram in Fig. 1. The magnitude of the road force is made a symbol since, at this point, it is unknown.
4. Write the force equation in terms of horizontal and vertical components. This gives two equations in two unknowns (the magnitude of the road force and the value of θ used to break the road force into components).
5. Solve for one or both of the unknowns, as desired.

3c. The Deduced Angle. The result of carrying out the steps outlined above gives for the proper banking angle:

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right), \tag{1}$$

where v is the speed of the vehicle, r is the radius of the turn, and g is the acceleration of gravity in free fall ($9.8\,\mathrm{m/s^2}$). Incidentally, the normal force N of the road on the car is:

$$N = \frac{mg}{\cos\theta} = \frac{\text{weight}}{\cos\theta}. \tag{2}$$

▷ Carry out the five steps listed above and deduce Eqs. (1) and (2). *Help: [S-8]*

▷ For the example at the end of Sect. 2c, show that the proper banking angle is 23°. *Help: [S-12]*

3d. Analysis of the Results. First, note that the ideal banking angle, θ, is independent of the vehicle's mass! Therefore the same banking angle can serve all vehicles provided they move with the design speed:

$$v = \sqrt{rg\tan\theta}; \qquad \textit{Help: [S-9]}$$

For somewhat larger or smaller speeds, the frictional force between tires and road will keep the vehicle on the curve if tire tread and road surface permit.

▷ Suppose the road has become icy during a blinding snowstorm and you head into a short-radius curve, steeply banked for 55 mph, at 25 mph (as I once did). What do you think will happen? Draw the one-body diagram!

4. Other Examples

4a. Whirling Rock on a String. For the whirling rock example of Sect. 2, we follow the procedure used in Sec. 3 and deduce the angle of the string. The result is:

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right), \tag{3}$$

where v is the speed of the rock, r is the radius of the circle, and g is the acceleration of gravity.

▷ Follow the referenced steps and deduce Eq. (3). *Help: [S-10]*

4b. Bicyclist on a Turn. To deduce the angle of lean, in the turning bicyclist of Sect. 2, we follow the procedure used above and in Sec. 3 to find:

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right), \tag{4}$$

where v is the speed of the bike (and rider), r is the radius of the turn-circle, and g is as usual.

▷ Follow the referenced steps and deduce Eq. (4). *Help: [S-11]*

4c. "Weight" on a Banked Turn. When a vehicle travels a banked turn, the driver's weight seems to increase. That is, the driver has the same feeling that would be experienced if the force of gravity were suddenly increased. Similarly, the vehicle's tires flatten more against the pavement, as though the car was also experiencing increased gravity. The driver's physical sensations are an increase of the force of the car seat on the driver's posterior and similar sensations in the driver's internal organs. This apparent increase in weight, due to the increased normal force, is given by Eq. (2).

▷ In the example at the end of Sect. 2c, show that the effect is as though the force of gravity had increased by about 9% while traversing the turn.

4d. Circular Motion and Weightlessness. A pilot in a plane can produce a temporary feeling of "weightlessness" by aiming the plane slightly upward and then making a tight turn downward. The plane thus moves in a circular arc in a vertical plane. At the peak of the plane's path, where it is instantaneously parallel to the earth's surface, the resultant force is in the same direction as the force of gravity. If, at that point, the speed and radius are such that $a = v^2/(r) = g$, then the resultant force exactly equals the force of gravity. Of course the resultant force on the pilot is the sum of the (downward) force of gravity and the (upward) force of the plane's seat on the pilot's posterior. In our present case the resultant force equals the force of gravity alone, so the force of the seat on the pilot must be zero and this is what accounts for the feeling of "weightlessness."

4e. Acceleration in g's. People who routinely engage in tight turns, such as fighter pilots, measure the accelerations they experience in g's, sometimes spelled "gees," which simply means they divide the acceleration in ordinary units by the quantity g, expressed in the same units. For example, consider the "weightless" turn described above. If the pilot continued in the same circle at the same speed, the point at the bottom of the circle would be a "one gee" turn. *Help: [S-13]* At this instant the pilot's weight would seem to have doubled. Similarly, a two gee turn causes an apparent tripling of body weight. At four to five gees, insufficient blood reaches the brain and the sitting pilot "blacks out."[2]

5. Force-Words for Circular Motion

5a. Centripetal Force. The radial acceleration acting in uniform circular motion, $a = v^2/r$, is frequently referred to as the "centripetal" acceleration. The *resultant* force that produces this acceleration is called the "centripetal" force, but note that this (resultant) force is almost always the sum of forces produced by various agents.

5b. Centrifugal Force. Newton's third law says that for every force there is an "equal but opposite" force, and the force "equal but opposite" to the centripetal force is called the "centrifugal" force. In the example of the airplane pilot at the bottom of a vertical circular turn, the (upward) centripetal force is the force of the seat on the pilot. The centrifugal force is the force of the pilot on the seat. Similarly, for a person in a horizontal centrifuge-type ride in an amusement park, the centripetal force is the radially inward force of the seat on the person, producing the observed

[2]Pilots can sustain up to 10 gees by staying there no more than seconds before returning to 2-3 gees for a short recovery period, and up to 15 gees in a reclining (rather than sitting) position.

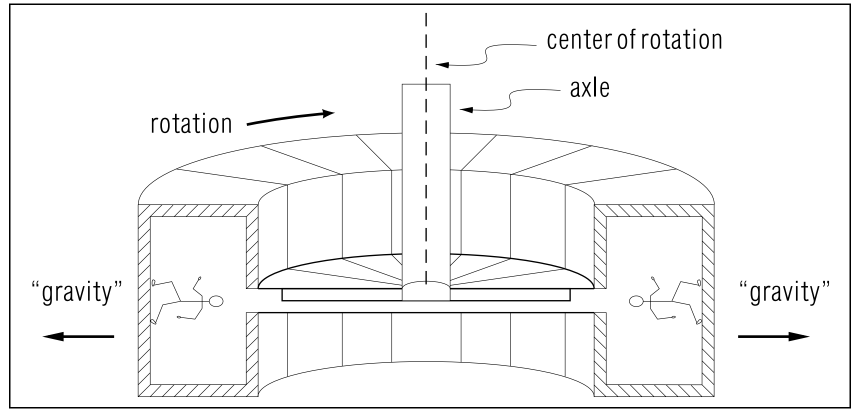


Figure 4. Cross-sectional sketch of a possible space station. Rotation causes simulation of gravity for each person along the outer rim.

acceleration, while the centrifugal force is the equal but opposite force of the person on the outside edge of the seat (if the seat is not strong enough to withstand the centrifugal force, there will be an accident).

In the case of a bicyclist making a sharp turn, the resultant force is exerted on the bicycle and rider by the road surface as a force of non-sliding friction. The centrifugal force is exerted by the bicycle on the ground surface. If the bicycle hits a loose pebble or a slippery spot, the coefficient of friction may suddenly drop to zero with disastrous results.

In a common classroom demonstration a student sits on a rotating lab stool, holding a weight in an outstretched arm. The student's hand exerts the centripetal force on the weight, producing its acceleration. The weight exerts the reactive centrifugal force on the student's hand, and of course this is what the student feels.

5c. The Rotating Space Station. Gravity can be simulated in a space station by causing it to rotate at the right speed. Consider the rotating space station shown in Fig. 4. The two persons shown in the sketch are upside-down with respect to each other, yet each exerts a centrifugal force on the floor underfoot. If the speed of rotation v is related to g and the radius r by $a = v^2/r = g$, and if each person stands on a scale, the centrifugal force each exerts on the local scale will be exactly equal to the person's own weight.

5d. Coriolis Force. Curving motion produces yet another force that can sometimes be important. Called the "Coriolis" force, it arises when one observes trajectories from a rotating frame of reference (usually the surface of the earth).[3] This is the force that causes the plane of the Foucault pendulum[4] to appear to rotate, both demonstrating the rotation of the earth and indicating the latitude of the pendulum. The Coriolis force also causes the regularity in the direction of rotation of the common extratropical cyclones seen on weather satellite photographs, and it is involved in the prevailing directions of the major winds and ocean currents. It can be demonstrated on a rotating lab stool.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **banking angle**: the angle at which the roadbed of a highway curve is tipped from the horizontal.
- **centrifugal force**: the reactive force to the centripetal force, to which it is equal but opposite.
- **centripetal acceleration**: the radially-inwardly component of acceleration for an object in circular motion.
- **centripetal force**: the radially-inward component of the resultant force on an object in circular motion.
- **ideal banking angle**: the banking angle such that, for an object at a particular speed undergoing circular motion, there is no "sideways" force on the object. For a highway turn, this means that the road surface exerts only a normal (perpendicular) force on vehicles traveling at the design speed. The resultant of the (normal) road force and the force of gravity is the centripetal force that causes the vehicle's velocity vector to constantly change direction as the vehicle travels through the turn.

[3]See "Classical Mechanics in Rotating Frames of Reference: Effects on the Surface of the Earth" (MISN-0-18).

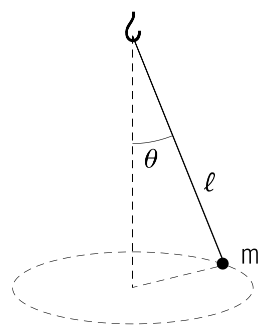
[4]This is the pendulum commonly seen in science museums, usually several stories tall, consisting of a long wire and a steel ball.

PROBLEM SUPPLEMENT

1. The space station in *2001: A Space Odyssey* is constructed in the shape of two wheels connected by an axle (see Fig. 4 in this module's *Text* for a drawing of one of the wheels and part of the connecting axle). If the "wheels" are each 300 m in diameter, what rotational period would be necessary in order to simulate the familiar force of gravity at the earth's surface?

2. As in Problem 1 but for the SKYLAB satellite: what rotational period would be necessary for this satellite, which is about 16 m in diameter?

3. A ball of mass m is attached to the end of a string of length ℓ; the other end is tied to the ceiling. The ball is set into motion in a circular path as shown.



   a. Make a one-body diagram for the ball.

   b. What must its speed be for the string to make a given angle θ with the vertical?

   c. Find the tension in the string.

4. You are rounding a turn of radius 0.500 mile (2,640 ft) at 175 mi/hr (257 ft/s) in the Indy 500. In the following, neglect effects due to rotation of the earth and air resistance.

   a. Derive the ideal track banking angle in symbols, then in degrees. Sketch a clear one-body force diagram showing all forces, plus the resultant, that act on the car.

   b. Explain why this banking angle is ideal.

   c. Calculate the force of the car on you, in multiples of your body weight.

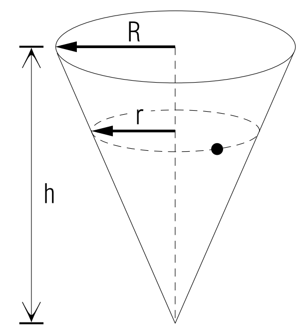
   The Indy 500 track was actually built in 1909 and banked for 100 mi/hr (147 ft/s).

   d. Calculate its actual banking angle. Sketch a clear one-body force diagram showing the resultant force for this case (100 mi/hr).

   e. Sketch a clear one-body force diagram showing the force needed to keep your 175 mi/hr car at constant radius, as well as the part of that force supplied by the horizontal component of the actual track's support. Also show the sideways force parallel to the track which must be supplied by static and skidding friction between tires and track. This is the difference between the above two forces.

   f. Identify the centripetal and centrifugal forces associated with your body, in the sense of stating what body is acting and what body is acted on in each case.

5. A ball of mass m rolls on the inside of the frictionless circular cone of height h and base of radius R. The axis of the cone is vertical, and the apex points down. That ball is set into motion in a horizontal circular path of radius r. ($0 < r < R$)



   a. Draw a one-body diagram for the ball.

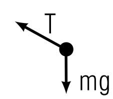
   b. What is the required speed v of the ball?

Brief Answers:

1. $T = 2\pi\sqrt{r/g} = \pi\sqrt{2d/g} = 25\,\mathrm{s}$; $d =$ diameter

2. $T = \pi\sqrt{2d/g} = 5.7\,\mathrm{s}$

3. a.



   b. $v = \sqrt{\ell\, g\, \sin^2\theta / \cos\theta}$ *Help: [S-1]*

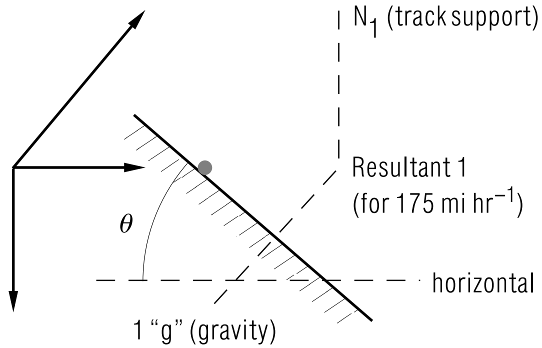
   c. $T = m\, g\, \sec\theta$ *Help: [S-2]*

4. a. $\tan\theta = v^2/(Rg)$

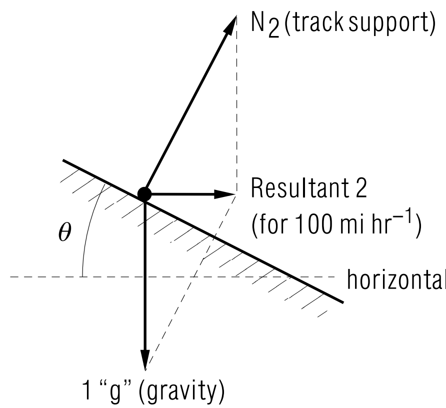
   $\theta = 38.02°$.

   b. Track must not exert a sideways force parallel to itself. Then there will be no tendency for a car to skid sideways.

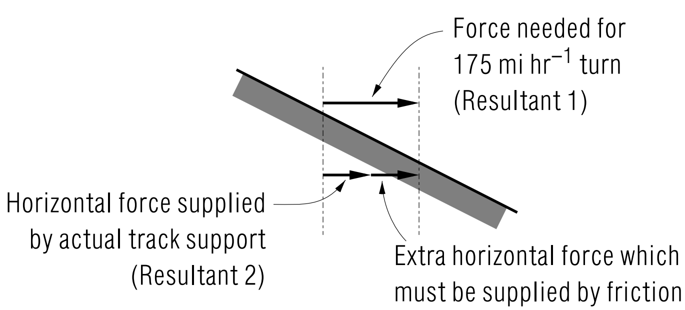
   c. 1.27 times your weight. *Help: [S-14]*



   d. $\theta_0 = 14.35°$.



   e.



   f. car door (or shoulder belt) on you, you on car door (or shoulder belt)

5. a. The force diagram looks similar to that of problem 3a but with the force T replaced by a force normal (perpendicular) to the frictionless surface.

   b. $\sqrt{rgh/R}$ *Help: [S-4]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 3b)*

1. Determine the actual acceleration the mass undergoes, in terms of given and desired quantities. Get rid of r by substituting $\ell \sin\theta$.
2. Multiply the acceleration by the mass to get the force acting on the mass.
3. Draw the force on the diagram in the problem statement.
4. Did you properly point the force toward the center of the circle?
5. Write the total force on the mass as $\vec{T}$, the string's (unknown) force, plus $m\vec{g}$: these are the only forces acting on the mass.
6. Set the force deduced from the acceleration equal to the force deduced from adding the acting forces. This produces a vector equation.
7. Rewrite the vector equation as two single-component equations (one for the x-components, one for the y-components), each involving θ.
8. Eliminate T between the two equations, leaving one equation.
9. Solve for v^2.

S-2 *(from PS-Problem 3c)*

$\sec\theta \equiv 1/\cos\theta$

S-4 *(from PS-Problem 5b)*

1. Completely solve problem 4 before attempting this one.
2. The surface of the cone is said to be "frictionless," so the cone surface only exerts a force normal to itself. That's the physics. Now using trigonometry, we find that the force of the surface on the ball is at an angle of $\tan^{-1}[R/h]$ from the horizontal.

S-7 *(from TX-2c)*

$$\frac{[(50\,\text{mi/hr})(5280\,\text{ft/mi})(1/3600\,\text{s/hr})]^2}{(390\,\text{ft})(32\,\text{ft/s}^2)} = 0.43$$

S-8 *(from TX-3c)*

Taking components as stated:

$$N\sin\theta = F_r = \frac{mv^2}{r}$$
$$N\cos\theta = mg$$

S-9 *(from TX-3d)*

Solve Eq. (1) for v.

S-10 *(from TX-4a)*

The equations are exactly the same as in [S-8] but with N replaced by T, the tension in the string.

S-11 *(from TX-4b)*

The equations are exactly the same as in [S-8].

S-12 *(from TX-3c)*

$$\tan^{-1}\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) = \tan^{-1} 0.43 = 23^\circ$$

Help: [S-7]

S-13 *(from TX-4e)*

The acceleration of the pilot, at that point, is $1\,g$ ($32\,\text{ft/s}^2$) upward. This is obviously a "1 gee" acceleration.

S-14 *(from PS-Problem 4c)*

$$\vec{F}_{\text{on you}} = \vec{F}_{\text{earth on you}} + \vec{F}_{\text{car on you}} = m_{\text{you}}\,\vec{a}_{\text{you}}$$

What *is* your acceleration?

MODEL EXAM

1. See Exam Skills K1-K2, this module's *ID Sheet*. The exam may include one or more of these skills, or none.

2. You are rounding a turn of radius 0.500 mile (2,640 ft) at 175 mi/hr (257 ft/s) in the Indy 500. In the following, neglect effects due to rotation of the earth and air resistance.

   a. Derive the ideal track banking angle in symbols, then in degrees. Sketch a clear one-body force diagram showing all forces, plus the resultant, that act on the car.

   b. Explain why this banking angle is ideal.

   c. Calculate the force of the car on you, in multiples of your body weight.

   The Indy 500 track was actually built in 1909 and banked for 100 mi/hr (147 ft/s).

   d. Calculate its actual banking angle. Sketch a clear one-body force diagram showing the resultant force for this case (100 mi/hr).

   e. Sketch a clear one-body force diagram showing the force needed to keep your 175 mi/hr car at constant radius, as well as the part of that force supplied by the horizontal component of the actual track's support. Also show the sideways force parallel to the track which must be supplied by static and skidding friction between tires and track. This is the difference between the above two forces.

   f. Identify the centripetal and centrifugal forces associated with your body, in the sense of stating what body is acting and what body is acted on in each case.

Brief Answers:

1. See this module's text.

2. See this module's *Problem Supplement*, Problem 4.