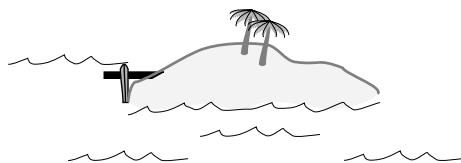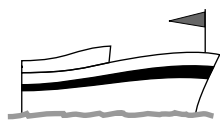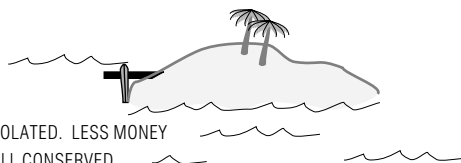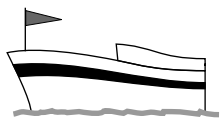PHYSNET

MISN-0-15

MOMENTUM:
CONSERVATION AND TRANSFER

1. ISOLATED ISLAND: NO MINT
   NUMBER OF DOLLARS IS CONSERVED
   (TOTAL NUMBER IS CONSTANT IN TIME)

2. CRUISE SHIP ARRIVES. NUMBER OF DOLLARS ON ISLAND
   INCREASES AS PASSENGERS MAKE PURCHASES. TOTAL
   DOLLARS OF SHIP AND ISLAND IS STILL CONSERVED

1. SHIP HAS LEFT. ISLAND ONCE AGAIN ISOLATED. LESS MONEY
   ON SHIP, MORE ON ISLAND. MONEY STILL CONSERVED.

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MOMENTUM: CONSERVATION AND TRANSFER
by
James M. Tanner, Georgia Institute of Technology

Title: **Momentum: Conservation and Transfer**

Author: James M. Tanner, School of Physics, Georgia Institute of Technology, Atlanta, GA

Version: 5/13/2002   Evaluation: Stage 1

Length: 1 hr; 40 pages

Input Skills:

1. State Newton's laws of motion and use them in the solution of one-particle dynamics problems where all forces are constant (MISN-0-14).
2. Use SI and English units in the solution of one-particle dynamics problems (MISN-0-14).
3. Add vectors graphically and formally (MISN-0-2).

Output Skills (Knowledge):

K1. Define the momentum of a particle and of a system of particles. State the SI and English units of momentum and express them in terms of force and time units.

K2. Using Newton's laws and the definition of momentum, show that the momentum of an isolated system of particles does not change with time.

K3. Show that conservation of momentum in collisions requires that each Cartesian component of momentum is conserved separately.

Output Skills (Problem Solving):

S1. Given an object's mass, its velocity before and after a collision, and data relevant to the collision time, calculate the average value of the resultant force on the object during the collision; and, given a collision time and the average resultant force on an object during the collision, calculate the object's change in momentum.

S2. Given information about two or more particles' momenta before and after a collision, use momentum conservation to determine quantities (speed, directions, etc.) related to the collision.

MOMENTUM:
CONSERVATION AND TRANSFER

by

James M. Tanner, Georgia Institute of Technology

1. Introduction

1a. Describing Conservation Principles. Imagine a self-sufficient group of people living and working in an environment somewhat isolated from the rest of society; say on a secluded South Pacific island (see this module's cover). The people have familiar homes, jobs, businesses, etc. Some time ago they adopted American currency as the medium of exchange. Lacking a formal government, they have no mints and have agreed to keep the total number of dollars fixed, except when they acquire a surplus of goods which are then sold to an American merchant ship that makes very infrequent calls. The captain of the ship always pays for the goods with American currency, thereby increasing the total number of dollars available to the people.

Between visits of the ship the total number of dollars in the society is fixed. Any money added to one person's dollar possession is always balanced by a decrease in another's. The important point here is that if at any instant the dollar possession of all inhabitants are totaled, the sum remains constant (until the visiting ship adds new money).

Physicists would describe the constraint on the total money available as a Conservation of Dollar Principle. The number of dollars any person possesses may change fifty times per day, but at any instant the sum of dollars summed over the societal system is fixed, or "conserved." Thus a conservation principle is nothing more than a statement that for a given system (e.g., the isolated community of people) there exists some quantity (total number of dollars) which remains constant in time provided certain conditions are met (no dollars added by visiting ships).

1b. Conservation of Mass. Perhaps the simplest conservation principle in classical physics is that of mass conservation. Essentially, the principle is this. If a system, A, is defined and then isolated from all other systems by not permitting a flow of mass into or out of system A, the mass conservation principle asserts that the sum of the masses of the constituent objects in system A remains constant even though the number of objects may be changing (two lead blocks may be melted and solidified

into one, a block of cheese may be cut into ten slices, etc.), and the mass of individual entities may change with time (water evaporating from a glass, frictional wear of rubber from a tire, etc.). Classically, if the total mass of the universe at any one instant is M, this would apparently be the total mass at all times even though subsystem masses may be changing at very rapid rates because of interaction between the subsystems.

1c. Whether Mass Is Really Conserved. The mass conservation principle just asserted seemed to scientists of the nineteenth century to be an accurate statement of natural law. In the first half of this century, careful investigation of phenomena like relativistic physics, fission, fusion, antimatter, etc., led to the conclusion that, in fact, the total mass of a uranium nucleus is greater than the combined mass of the particles which result from its fission. As a result of these new discoveries and the accompanying interpretations, physicists concluded that mass, rather than being conserved, is just one form of energy, a quantity which is conserved.[1] What now? Are you to assume (in working a problem in this course) that mass is conserved or not? The answer is "Yes," but with the proviso that there are phenomena in which mass is not conserved. Your first experience with such phenomena will probably come in studying either relativistic or atomic physics or chemistry.[2]

1d. Conservation of Momentum. The topic of fundamental importance in this unit is that of momentum conservation. You will see that for a system of particles which are isolated from interaction with other particles, a vector quantity called the system's "total (linear) momentum" is constant in time. This will be true even though the system's individual constituent particles may have rapidly changing momenta. This situation is not too different from the fixed number of dollars available to the isolated island dwellers whose individual dollar possessions may change rapidly as a result of interaction with other island dwellers.

1e. Newton's Laws and Conservation of Momentum. In Sect. 4 we will use Newton's laws to derive the momentum conservation principle. This shows that the momentum conservation principle is valid throughout Newtonian dynamics. While Newtonian dynamics is valid over a very

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[1] See "Potential Energy, Conservative Forces, The Law of Conservation of Energy" (MISN-0-21), and "Work, Kinetic Energy, Work-Energy Principle, Power," (MISN-0-20). "Collisions" (MISN-0-32) illustrates the usefulness of the two conservation principles (energy and momentum)

[2] See "Relativistic Energy; Thresholds for Reactions" (MISN-0-23).

wide range of natural phenomena,[3] radical corrections can be necessary for high speeds and/or high masses, and for phenomena on the atomic scale. However, the momentum conservation principle is valid for all of those phenomena as well as for Newtonian dynamics. There are no known data which contradict it: conservation of momentum is one of the few universal physical principles known.

2. Momentum and Force

2a. Definition of Momentum.   The linear momentum $\vec{p}$ associated with a particle of mass m moving with a velocity $\vec{v}$ is defined by:

$$\vec{p} \equiv m\,\vec{v}.$$

As a particle moves, its velocity changes as a result of its interaction with other particles, so we can write:

$$\vec{p}(t) \equiv m\,\vec{v}(t) \tag{1}$$

2b. An Isolated Particle has a Constant Momentum.   First, let us consider the special case of an isolated particle, one having no measurable interaction with other particles. This case is easy since, by Newton's first law, such a particle moves with a constant velocity.[4] Since the mass is constant, it thereby moves with a constant momentum:

$$\vec{p} = \text{constant.} \qquad \text{(Isolated Particle)} \tag{2}$$

This is the purest and simplest statement of the Conservation of Momentum Principle. Unfortunately, in this form it is also the most useless statement. Do not despair, for in Sect. 4 you will see how to enrich this principle to make it useful.

2c. Force Equals Time-Rate-of-Change of Momentum.   Before extending the conservation discussion, let's investigate how a particle's momentum changes. To do so, differentiate each side of Eq. (1) with

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[3] See "Particle Dynamics, Mass, Force, and the Laws of Motion" (MISN-0-14), Section 5b.

[4] See "Particle Dynamics: The Laws of Motion" (MISN-0-14).

respect to time:

$$
\begin{aligned}
\frac{d\vec{p}}{dt} &= \frac{d}{dt}(m\vec{v}) \\
&= m\frac{d\vec{v}}{dt} \qquad (m \text{ is constant}) \\
&= m\vec{a}. \qquad \left(\vec{a} \equiv \frac{d\vec{v}}{dt} : \text{ Definition}\right)
\end{aligned}
$$

Now, according to Newton's second law $m\vec{a}$ equals the resultant force $\vec{F}_R$ acting on the particle so:

$$\frac{d\vec{p}}{dt} = \vec{F}_R. \tag{3}$$

Equation (3) is another way to express Newton's second law ($\vec{F}_R$ is the vector sum of all forces acting on the particle whose momentum is $\vec{p}$).

2d. Which is More Fundamental, Force or Momentum.   Equation (3) can be given at least two interesting interpretations. If you prefer to think of force as being that which causes acceleration, then you can interpret $\vec{F}_R = d\vec{p}/dt$ as stating: a resultant force being exerted on the object we are studying implies that the object's momentum must be changing. If you prefer to think of "the natural state" of an isolated particle being one of constant momentum, then you can think of $\vec{F}_R = d\vec{p}/dt$ as stating: a changing momentum implies that other objects must be exerting a resultant force on the object we are studying.

Another interesting facet of momentum is that it plays a crucial role in atomic physics, where the concept of force does not exist.

Enough of this patter about what fundamental meaning (if any) is provided by this new way of writing Newton's second law. The crucial information for you at this point is that the resultant force on a particle equals the time-rate-of-change of that particle's momentum [Eq. (3)].

3. Collision Forces

3a. Examining the Force Exerted on a Ball by a Bat.   Situation: Nolan Ryan (California Angel pitcher) fires his fastest pitch (approximately 100 mi/hr or 30 m/s) to Henry Aaron (Milwaukee Brewer designated hitter) who swings and sends a hit (at the same speed) past Nolan's head.
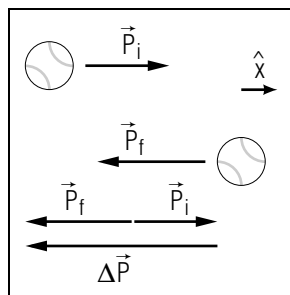
Figure 1. The momentum change during the collision.

Question: If the magnitude of the ball's weight is W, the average force of the bat on the ball during contact is most nearly equal to:

(a) W (c) $100\,W$ (e) $10,000\,W$
(b) $10\,W$ (d) $1000\,W$ (f) $100,000\,W$

▷ What's your best guess?

3b. If the Force Were Constant, Here's the Answer. In order to obtain an approximate answer for the force on a ball during collision with a bat, we assume that the force on the ball is constant during the collision. This seems to be a mediocre approximation at best, since intuitively you probably feel that the force at the times of initial and final contact is much less than that when one of the ball's sides has "pancaked" against the bat and is momentarily at rest with respect to it. Nevertheless, we'll begin with the "constant force during the collision" assumption in order to make things simple. We start with:

$$\frac{d\vec{p}}{dt} = \vec{F}.$$

We multiply each side by Δt, the collision time, to get:

$$\frac{d\vec{p}}{dt}\Delta t = \vec{F}\Delta t.$$

Since $\vec{F}$, and so also $d\vec{p}/dt$, is constant in time, the quantity $(d\vec{p}/dt)\Delta t$ is the change in the ball's momentum, $\Delta\vec{p}$, during the time of the collision. So now we have:

$$\Delta\vec{p} = \vec{F}\Delta t;$$
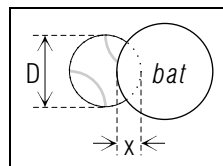
Figure 2. The collision-deformed ball.

or:

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}. \qquad \text{(Constant Force)} \qquad (4)$$

To find the force on the ball, using Eq. (4), we must compute the ball's change in momentum, $\Delta\vec{p}$, during the time interval Δt. Consider Fig. 1, showing the initial and final momenta. Note that the momentum of the ball is reversed in direction by the collision but is left unchanged in magnitude. Then:

$$\begin{aligned} \Delta\vec{p} &= \vec{p}_f - \vec{p}_i \\ &= -mv_0\hat{x} - (mv_0\hat{x}) \\ &= -2mv_0\hat{x}, \end{aligned}$$

where v_0 is the ball's speed ($30\,\text{m/s}$ in this case) and $\hat{x}$ is a unit vector in the direction of motion before the collision.

Now what about Δt, the time for collision? No doubt you have seen strobe flash pictures of a ball and bat in contact which show the ball to be distorted, as represented in Fig. 2. We will use what we have seen in such pictures in order to make estimates.

Let's estimate the maximum value of x (see Fig. 2) to be about one-fourth of the ball's diameter D. This maximum compression occurs at the instant the ball is at rest with respect to the bat (at approximately one-half Δt). During this time, the first half of the collision time, the ball's speed has gone from v_0 to zero (neglecting the bat's speed). Assuming constant deceleration, the ball's average speed is $v_0/2$. Now (av. speed)·(time) $\simeq$ distance, so:

$$\frac{v_0}{2} \cdot \frac{\Delta t}{2} \simeq \frac{D}{4};$$

or:

$$\Delta t \simeq \frac{D}{v_0}.$$

With these estimates for $\Delta\vec{p}$ and Δt, we use Eq. (4) to get

$$\vec{F} \simeq \frac{-2mv_0^2\hat{x}}{D}$$

or

$$F \simeq \left(\frac{2v_0^2}{gD}\right) mg = \left(\frac{2v_0^2}{gD}\right) W, \qquad (5)$$

where $W = mg$ is the ball's weight. Substituting approximate values for $v_0 \simeq 30\,\text{m/s}$, $g \simeq 10\,\text{m/s}^2$, and $D = 0.10\,\text{m}$, gives:

$$F \simeq \left(\frac{2 \times 900\,\text{m}^2/\,\text{s}^2}{10\ \text{m/s}^2 \times 0.10\,\text{m}}\right) W = 1800W.$$

So, if your answer in the original problem was (d), $1000\,W$, congratulations!

▷ By the way, how many pounds or Newtons is that?

3c. But the Collision Force Isn't Constant. Now let's see what we can do about the "constant force" swindle of the previous paragraph. Starting with $\vec{F}(t) = d\vec{p}/dt$, where we have explicitly noted the dependence of $\vec{F}$ on t, we integrate both sides of the equation with respect to time, from $t = 0$ (ball first touches bat) to $t = \Delta t$ (ball leaves bat):

$$\int_0^{\Delta t} \vec{F}_R(t)\,dt = \int_0^{\Delta t} \frac{d\vec{p}}{dt}\,dt.$$

But

$$\int_0^{\Delta t} \frac{d\vec{p}}{dt}\,dt = \vec{p}(\Delta t) - p(0) = \Delta\vec{p}.$$

Then:

$$\int_0^{\Delta t} \vec{F}_R(t)\,dt = \Delta\vec{p}. \qquad (6)$$

In the Appendix we show that an integral of a vector function of a single scalar variable, such as the one in Eq. (6), can be written as the average value of the function $\vec{F}_{R,av}$ multiplied by the averaging interval Δt. So:

$$\vec{F}_{R,av}\Delta t = \Delta\vec{p};$$

or:

$$\vec{F}_{R,av} = \frac{\Delta\vec{p}}{\Delta t}. \qquad (7)$$
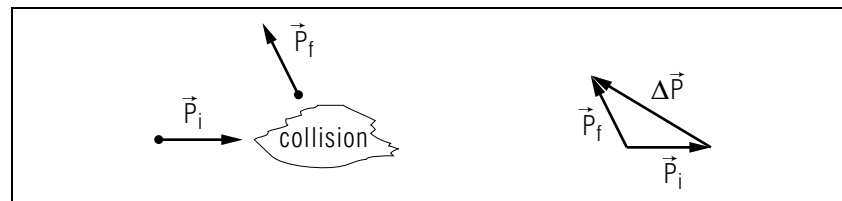
Figure 3. A collision in which a particle's momentum changes from $\vec{p}_i$ to $\vec{p}_f$.

Compare Eqs. (4) and (7). The time-average resultant force over a time interval Δt is exactly equal to the constant force which would cause that same momentum change during that same time interval. Consequently, our previous estimate of approximately 1800 times the ball's weight is exactly correct for the average resultant force on the ball.

3d. Why Collision Forces Are Usually So Large. While we obtained Eq. (7) in considering a ball-bat collision, you can discover by rereading that its derivation was completely general. Thus the time-average resultant force on an object always equals its change in momentum divided by the time interval during which the momentum change occurred: $\vec{F}_{R,av} = \Delta\vec{p}/\Delta t$ (see Fig. 3). Thus for a given momentum change, the average force increases as the collision time decreases. And that's why pole vaulters land on a few feet of foam rubber rather than concrete! In each case, their momentum change would be the same, but the increased collision time resulting from the foam's compressibility saves the day ... not to mention the pole vaulter.

3e. The Impulse-Momentum Principle. The impulse of a force $\vec{F}(t)$ during a time interval $t_1 \leq t \leq t_2$ is defined as:

$$\text{Impulse} \equiv \int_{t_1}^{t_2} \vec{F}(t)\,dt.$$

Comparing this with Eq. (6), we have the so-called Impulse-Momentum Principle:

Impulse by resultant force during time interval Δt	=	Change in momentum during time interval Δt

3f. Solution Using Components. Here we will illustrate solving a two-dimensional average-force problem using Cartesian compo-

nents (a more elegant method is developed in the Problem Supplement).
Problem: A car having mass m, sliding on ice at speed v^i, hits a concrete wall and bounces off, as shown, with speed v^f. If the collision time is measured from high speed movies to be Δt, determine the average force (magnitude and direction) exerted on the car by the wall.
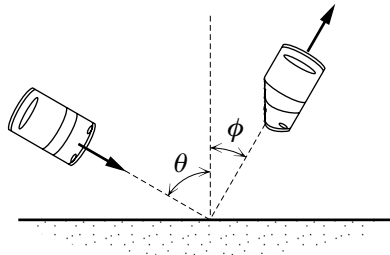
We start with Eq. (7), written in terms of its Cartesian components:

$$\vec{F}_{R,av} = \frac{\Delta p_x}{\Delta t}\hat{x} + \frac{\Delta p_y}{\Delta t}\hat{y} + \frac{\Delta p_z}{\Delta t}\hat{z} \,. \tag{8}$$

We take the scalar product of $\hat{x}$ with each term in Eq. (8) and we get the x-component of the average resultant force:

$$F_{R,av,x} = \frac{\Delta p_x}{\Delta t} \,, \tag{9}$$

and of course there are similar equations for the y- and z-components of $\vec{F}_{R,av}$, obtained by mutiplying Eq. (7) by $\hat{y}$ or $\hat{z}$. Putting in the given quantities for the x- and y-components of the car's initial momentum $\vec{p}^{\,i}$ and final momentum $\vec{p}^{\,f}$:

$$\Delta p_x = p_x^f - p_x^i = mv_x^f - mv_x^i = mv^f \sin\phi - mv^i \sin\theta \,,$$

$$\Delta p_y = p_y^f - p_y^i = mv_y^f - mv_y^i = mv^f \cos\phi + mv^i \cos\theta \,.$$

where we have taken the downward and rightward directions as positive. Then:

$$F_{R,av,x} = \frac{mv^f \sin\phi - mv^i \sin\theta}{\Delta t} \,,$$

$$F_{R,av,y} = \frac{mv^f \cos\phi + mv^i \cos\phi}{\Delta t} \,.$$

angle CCW from upward vertical $= \tan^{-1}(F_{R,av,x}/F_{R,av,y})$.

▷ Suppose the car weighs 3200 lb, has an initial speed of 41 mi/hr (60.1 ft/s), a final speed of 22 mi/hr (32.3 ft/s), and a collision time of 0.50 s. Show that the average force (magnitude and direction) exerted on the car by the wall during that time interval is 1.4×10^4 lb at 32°CCW from the upward direction or 122°CCW from the rightward direction.
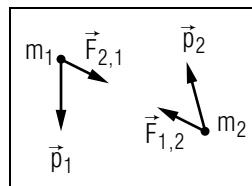
Figure 4. An isolated two-particle system.

4. Conservation of Momentum

4a. Dealing With Larger Systems. For a single-particle system, the conservation of momentum principle is equivalent to Newton's first law (See Sect. 2b). We will now extend the discussion to systems with two or more particles and discover that the momentum conservation principle acquires added significance.

4b. Derivation for Isolated Two-Particle Systems. Using Newton's second law and third laws, we can derive conservation of momentum for any isolated system consisting of two interacting particles. This is a system in which two particles interact with each other but not with other particles. The system momentum at the time t is denoted by $\vec{P}(t)$ and defined by

$$\vec{P}(t) = \vec{p}_1(t) + \vec{p}_2(t), \tag{10}$$

where $\vec{p}_n(t)$ is the momentum of particle n at time t (See Fig. 4). Differentiating both sides of Eq. (10) with respect to time, and using Newton's second law, we get:

$$\frac{d\vec{P}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \vec{F}_{R1}(t) + \vec{F}_{R2}(t), \tag{11}$$

where $\vec{F}_{Rn}(t)$ is the resultant force on particle n at time t. Since the system is assumed to be isolated, particle 1 interacts only with particle 2 so:

$$\vec{F}_{R1}(t) = \vec{F}_{2 \text{ on } 1}(t),$$

where $\vec{F}_{2 \text{ on } 1}(t)$ is the force particle 2 exerts on particle 1 at time t. Similarly,

$$\vec{F}_{R2}(t) = \vec{F}_{1 \text{ on } 2}(t).$$

Now by Newton's third law, $\vec{F}_{1 \text{ on } 2}(t)$ and $\vec{F}_{2 \text{ on } 1}(t)$ are equal in magnitude and opposite in direction at each instant of time. Then Eq. (11) becomes:

$$\frac{d\vec{P}}{dt} = \vec{F}_{2 \text{ on } 1}(t) + \vec{F}_{1 \text{ on } 2}(t) = 0.$$

Since its time rate of change is zero, the total linear momentum for an isolated two-particle system must not change with time; i.e.,

$$\vec{P}(t) = \text{Constant.} \qquad \text{(Isolated System)} \qquad (12)$$

Very interesting. While $\vec{p}_1$ and $\vec{p}_2$ may be changing very rapidly because of the two particles' mutual interaction, their sum $\vec{P}$ remains constant. In simple terms, $\vec{P}$ represents the total momentum available to the two particles (as long as they are isolated), and the two of them "swap" parts of that momentum back and forth via their interaction.

4c. The General Conservation of Momentum Principle. Here is the principal result of this unit:

Conservation of Momentum Principle	The total linear momentum for an isolated system of interacting particles is constant.

Thus even for 10^{20} interacting particles, each of which may have an erratically changing momentum vector, the sum of these 10^{20} vectors will remain constant provided the total system is isolated from any other system. The proof of this principle, for Newtonian mechanics, is relegated to Problem 2 of this module's *Problem Supplement.*

4d. When the System Isn't Isolated. If a system is not isolated, then the rate of change of the system's total linear momentum equals the sum of the resultant external forces acting on all the particles of the system. We can easily demonstrate this for a two-particle system, starting with Eq. (11):

$$\frac{d\vec{P}}{dt} = \vec{F}_{R1}(t) + \vec{F}_{R2}(t).$$

If particles 1 and 2 interact with other particles as well as each other, then the resultant force on each is the sum of the force on it by the other particle and the force on it by particles external to the system; i.e.,

$$\vec{F}_{R1} = \vec{F}_{2\text{ on }1} + \vec{F}_1^{\text{ext}}$$

$$\vec{F}_{R2} = \vec{F}_{1\text{ on }2} + \vec{F}_2^{\text{ext}}.$$

Substituting these into the equation for $d\vec{P}/dt$ gives

$$\frac{d\vec{P}}{dt} = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}},$$

where the internal forces cancel once again because of Newton's third law. Now let $\vec{F}_R^{\text{ext}}$ be the sum of the external force on the system; i.e.,

$$\vec{F}_R^{\text{ext}}(t) = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}},$$

so that

$$\frac{d\vec{P}}{dt} = \vec{F}_R^{\text{ext}}. \qquad (13)$$

Comparing this result with the corresponding equation for a one-particle system, $d\vec{p}/dt = \vec{F}_R$, you see that the momentum for the two-particle system changes just as if the system were a single particle acted upon by the external force $\vec{F}_R^{\text{ext}}$.

5. Collision Momenta

5a. Using Momentum Conservation to Analyze Collisions. You will find momentum conservation useful in many applications, particularly in phenomena which involve the collision of two or more particles. In these collisions the particles usually undergo rapid momentum changes which involve very large, but short-lived forces (remember the baseball). Even though the particles in such a collision may not be completely isolated from the surroundings, the collision forces are usually so large and the collision time so short that the system can be very accurately approximated as being isolated for the duration of the collision.

5b. Conservation of Momentum Components in Collisions. Each Cartesian component of momentum is separately conserved in momentum-conserving interactions and this fact is often used in solving two-dimensional and three-dimensional collision problems. The proof of momentum component conservation starts with the statement of conservation of momentum,

$$\vec{p}^{\,f} = \vec{p}^{\,i}, \qquad (14)$$

where $\vec{p}^{\,f}$ is the momentum of the system being studied at some "final" time, and $\vec{p}^{\,i}$ is the momentum of the system at some "initial" time. Equation (14) just says that the system's momentum does not change with time; that it is "conserved." We now expand each momentum vector in Eq. (14) in its Cartesian components:

$$p_x^f\,\hat{x} + p_y^f\,\hat{y} + p_z^f\,\hat{z} = p_x^i\,\hat{x} + p_y^i\,\hat{y} + p_z^i\,\hat{z}\,. \qquad (15)$$

Equation (15) can be written more succinctly as:

$$\sum_{n=1}^{n=3} p_n^f \, \hat{x}_n = \sum_{n=1}^{n=3} p_n^i \, \hat{x}_n \,, \tag{16}$$

where $x_1 \equiv x$, $x_2 \equiv y$, and $x_3 \equiv z$. Finally, we multiply Eq. (15) or Eq. (16) by any one of the three unit vectors and we get conservation of momentum for that component of momentum. For example, taking the scalar ("dot") product of Eq. (15) and $\hat{x}$ gives conservation of the x-component of momentum:

$$p_x^f = p_x^i \,. \tag{17}$$

To complete the proof, one need only say "and the same for the y- and z-components." More succinctly, taking the dot product of Eq. (16) and the nth unit vector, $\hat{x}_n$, gives conservation of the nth component of momentum:

$$p_n^f = p_n^i; \qquad n = 1, 2, 3 \,. \tag{18}$$

In collisions, each momentum in Eq. (14) is the sum of the momenta of the individual particles participating in the collision. Thus one must equate the sum of the x-components of the particles in the system before the collision to the sum of the x-components of the particles in the system after the collision. Then one must do the same for the y-components and perhaps the z-components. This means there will be several equations that one solves for the several unknowns in the problem.

5c. Conserved Quantities in Collisions. Momentum if conserved in any isolated interaction, but other quantities are also conserved and they place additional restrictions on what can happen during the interaction. Whether any particular restriction applies to a specific interaction depends on what particles are participating in the interaction. For example, if the system is isolated from outside forces then momentum must be conserved. As another example, if only "conservative forces" participate in the interaction then energy must be conserved.[5] There are many such quantities that can be conserved in certain classes of interaction. For this module the problems you are to solve will not assume energy conservation.

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[5]For the definition of *conservative forces*, see *Potential Energy, Conservation of Energy*, MISN-0-21.

5d. Example: Momentum Components in a Collision.
Problem: A mass m moving with a speed u collides with a mass M at rest. After the collision, each moves as shown. Assuming momentum conservation, we want to determine the speeds w and V after the collision.
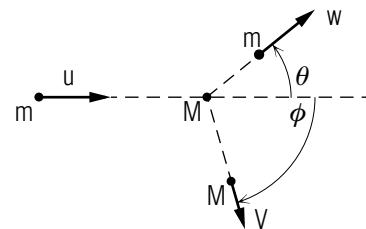
Adopting the notation introduced in Sect. 5b, and adding superscripts to identify the individual particles, the momentum x-components of the individual particles are:

$$p_x^{m,i} = mu; \quad p_x^{m,f} = mw\cos\theta; \quad p_x^{M,i} = 0; \quad p_x^{M,f} = MV\cos\phi \,,$$

while the y-components are:

$$p_y^{m,i} = 0; \quad p_y^{m,f} = mw\sin\theta; \quad p_y^{M,i} = 0; \quad p_y^{M,f} = -MV\sin\phi \,.$$

The system of two particles is assumed to have no external forces acting on it during the collision, so the system momentum components are conserved:[6]

$$p_x^{m,i} + p_x^{M,i} = p_x^{m,f} + p_x^{M,f} \,,$$
$$p_y^{m,i} + p_y^{M,i} = p_y^{m,f} + p_y^{M,f} \,.$$

For our case, these equations become:

$$mu = mw\cos\theta + MV\cos\phi \,,$$
$$0 = mw\sin\theta - MV\sin\phi \,. \tag{19}$$

Note that in this particular collision the system y-component of momentum is zero both before and after the collision. The two Eqs. (19) can be solved simultaneously for the two unknowns, w and V, in terms of the given quantities (m, M, u, θ, ϕ). It is just a little algebra.[7]

▷ Show that: $w = \dfrac{u\sin\phi}{\sin(\theta+\phi)}$; and: $V = \dfrac{m\,u\sin\theta}{M\,\sin(\theta+\phi)}$.

---

[6]Adding a requirement of energy conservation would add another equation and that would make it three relationships between two unknowns. No solution exists unless we allow one of the known variables to be an unknown variable, to be fixed by solving the three equations for the (then) three unknowns. One usually picks one of the outgoing angles, in this case θ or ϕ, which is then restricted by the known variables to be a single value.

[7]We also like to make the substitution: $\sin\theta\cos\phi + \cos\theta\sin\phi = \sin(\theta+\phi)$. This makes the answer more succinct but it is not really necessary.
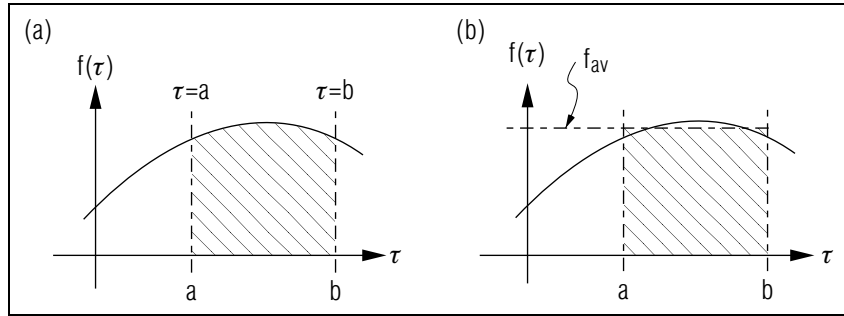
Figure 5. (a) $\int_a^b f(\tau)\,d\tau$ equals the shaded area. (b) f_{av} equals the height of the rectangle with area equal to $\int_a^b f(\tau)\,d\tau$.

Acknowledgments

A. Averaging a Vector Function

(for those interested)

If $\vec{C}(\tau)$ is a vector function of a single variable τ (e.g., position as a function of time), the integral of $\vec{C}$ on the interval $a < \tau < b$ is defined in terms of the corresponding integrals of the components of $\vec{C}$; i.e.,

$$\int_a^b \vec{C}(\tau)\,d\tau = \hat{x}\int_a^b C_x(\tau)\,d\tau + \hat{y}\int_a^b C_y(\tau)\,d\tau + \hat{z}\int_a^b C_z(\tau)\,d\tau. \quad (20)$$

Each of these three integrals are just integrals of a real function of one variable of the form

$$\int_a^b f(\tau)\,d\tau.$$

Remember that this integral is the area of the region inside $y = f(\tau)$, $\tau = a$, $\tau = b$, and the x-axis. This area is the shaded region in Fig. 5a. The average value of f on this interval, denoted by f_{av}, equals the height

of the rectangle of the same area as that defined by the integral. Thus

$$\text{Area of Rectangle } = \int_a^b f(\tau)\,d\tau.$$

But the area of the rectangle of height f_{av} and base $b - a$ is $f_{av}(b - a)$, so that

$$f_{av}(b - a) = \int_a^b f(\tau)\,d\tau,$$

or:

$$f_{av} = \frac{1}{b - a}\int_a^b f(\tau)\,d\tau.$$

This value is the dashed line in Fig. 5b. Note that it checks visually with what would expect for an "average" value.

Using the results we just obtained for a real function of one variable, we write for the vector function:

$$C_{x,av}(b - a) = \int_a^b C_x(\tau)\,d\tau,$$

$$C_{y,av}(b - a) = \int_a^b C_y(\tau)\,d\tau,$$

$$C_{z,av}(b - a) = \int_a^b C_z(\tau)\,d\tau.$$

Substituting these into Eq. (20), gives

$$\int_a^b \vec{C}(\tau)\,d\tau = (C_{x,av}\hat{x} + C_{y,av}\hat{y} + C_{z,av}\hat{z})\,(b - a).$$

We define the average of $\vec{C}(\tau)$ on $a < \tau < b$ as

$$\vec{C}_{av} \equiv C_{x,av}\hat{x} + C_{y,av}\hat{y} + C_{z,av}\hat{z} \qquad \text{(Definition)}.$$

Equation (13) may now be rewritten:

$$\vec{C}_{av} = \frac{1}{b - a}\int_a^b \vec{C}(\tau)\,d\tau.$$

PROBLEM SUPPLEMENT

This supplement has two distinct parts.

Part I contains four problems, each with a problem statement followed by a set of questions about the problem. Each set of questions is followed by a set of answers. Work through the set of questions for a problem, then compare your answers to the ones given. If they agree, great! If they don't, make sure that you reconcile them.

Part II of this supplement contains problems with less detailed questions and answers.

PART I

PART II

PART I

## A. One-dimensional Collision-Force Calculation.
Problem: Estimate the average resultant force on a paratrooper (mass = 79 kg) who hits the ground falling vertically at a speed of 11 m/s (25 mi/hr). Assume that he is a super-trooper and does not fall when he lands, but just lets his knees bend to increase the collision time.

1. The average resultant force on the trooper is $\Delta\vec{p}/\Delta t$, where $\Delta\vec{p}$ is his change in momentum while stopping and Δt is _____.
   Answer: 1

2. $\Delta\vec{p} =$ _____. *Answer: 2*

3. Give a rough estimate for Δt. $\Delta t \simeq$ _____. *Answer: 3*

4. $\vec{F}_{R,av}$ _____. *Answer: 4*

5. How does this force compare to his weight? $F/mg =$ _____.
   Answer: 5

B. Two-dimensional Collision-Force Calculation.
Problem: A 3200 lb car sliding on ice at 41 mi/hr (60.1 ft/s) hits a concrete wall and bounces off, as shown, with a speed of 22 mi/hr (32.3 ft/s). If the collision time is measured from high speed movies to be 0.50 s, determine the average force (magnitude and direction) on the car by the wall.
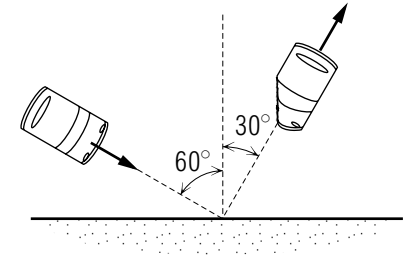
1. Since $\sum \vec{F}_{av} = \Delta\vec{p}/\Delta t$, you must determine $\Delta\vec{p}$. To do so, draw a vector triangle showing the car's initial momentum, final momentum, and change in momentum. *Answer: 6*

2. Determine the magnitude of the momentum change. Hint: Pay close attention to the vector triangle just constructed. $|\Delta\vec{p}| =$ _____. *Answer: 7*

3. Determine the angle between $\Delta\vec{p}$ and the normal to the wall. Angle = _____. *Answer: 8*

4. Determine the magnitude of the average force on the car by the wall. $|\sum \vec{F}_{av}| =$ _____. *Answer: 9*

5. Show, on the diagram, the average force on the wall by the car. Indicate its magnitude and direction. *Answer: 10*

## C. One-dimensional Conservation of Momentum Problem.
Problem: A mass m moving with a speed w to the right collides head-on with a mass M moving with a speed V to the left. After the collision, the two masses remain together. Determine the velocity u of the composite mass in terms of m, M, w, V, and $\hat{x}$, a unit vector to the right.

1. Read the problem carefully. Draw before and after sketches that depict the problem. Label the various masses and velocities with appropriate symbols. *Answer: 11*

2. Is it appropriate to assume the momentum of the two-particle system to be conserved during the collision? What assumptions, if any, must be made so that you can use momentum conservation? *Answer: 12*

3. Assuming momentum to be conserved, write a vector expression which relates m, M, w, V, $\hat{x}$, and $\vec{u}$. *Answer: 13*

4. Solve for $\vec{u}$. What can you say about the direction for $\vec{u}$? $\vec{u} = $ _____. *Answer: 14*

5. If the magnitude of the momentum of m before the collision is _____than that of M, the composite particle moves to the right. If the two initial momenta are _____, the composite particle remains at rest; and if the initial momentum of m is _____that of M, the final motion is to the left. *Answer: 15*

6. Consider the special case for M initially at rest; i.e., $V = 0$. Show how a measurement of the initial and final speeds (w,u) could be used to determine the ratio of the two masses. *Answer: 16*

7. For the mass determination just discussed, suppose the data shows that $u \ll w$. Which inequality is appropriate: $M \ll m$ or $M \gg m$? *Answer: 17*

8. If you did not do so already, write a brief discussion explaining these results by considering the known results of collisions between small and large objects. *Answer: 18*

D. Two-dimensional Conservation of Momentum Principle.
Problem: A mass m moving with a speed u collides with a mass M at rest. After the collision, each moves as shown. Determine the speeds after the collision, if $m = 0.15\,\text{kg}$, $M = 0.20\,\text{kg}$, $\theta = 39°$, $\phi = 74°$, $u = 20.0\,\text{m/s}$.
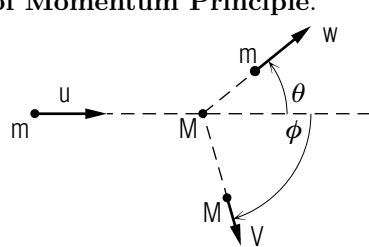
1. Read the problem carefully. Consider the two particle system consisting of m and M. Is this an isolated system? Why? *Answer: 19*

2. If you use momentum conservation to relate the known and unknown quantities, what assumptions about the system are you making? *Answer: 20*

3. Under what condition is this assumption valid, even if the system is not isolated? *Answer: 21*

4. Construct a vector diagram illustrating the momentum conservation equation: $m\vec{u} = m\vec{w} + M\vec{V}$. *Answer: 22*

5. The task now is to use this vector diagram to solve for w and V in terms of the known quantities m, M, u, θ, ϕ. Solutions are easily obtained by using the law of sines. Write out this relationship for the triangle shown. *Answer: 23*
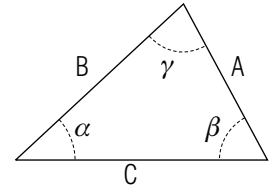
6. Now use this result to solve for w. Hint: Recall that $\sin(\pi - \alpha) = \sin\alpha$ for any value of α. *Answer: 24*

7. Solve for V. *Answer: 25*

8. You could have solved for w and V by writing down conservation of momentum equations. Using the momentum conservation triangle, write out the two corresponding scalar equations:

   components parallel to initial momentum: $mu = ?$

   components perpendicular to initial momentum: $0 = ?$ *Answer: 26*

9. Use the second equation to solve for MV in terms of m, w, θ, ϕ. Substitute this into the first, and solve the resulting equation for w. Your answer will simplify if you use the trig identity:

   $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$. *Answer: 27*
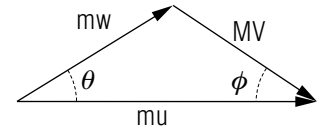
10. Now complete the solution by substituting the expression for w into the equation for V. *Answer: 28*

11. Note that the expressions for w and V obtained by two alternate means agrees. Now determine the values for w and V. *Answer: 29*

12. Notice that the vector triangle drawn to display conservation of momentum graphically was very important in solving this problem. In these types of problems, you will often find that the law of sines or law of cosines from trigonometry is a time-saver over the component method.
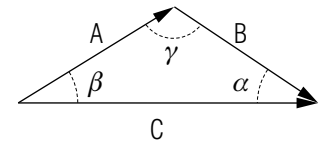
   These trigonometric relations are summarized in the figure and the following equations. The laws are applied to each vertex in turn.

Law of Sines

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta} = \frac{C}{\sin \gamma}$$

Law of Cosines

$$C^2 = A^2 + B^2 - 2AB \cos \gamma$$
$$B^2 = A^2 + C^2 - 2AC \cos \beta$$
$$A^2 = B^2 + C^2 - 2BC \cos \alpha$$

PART II

> **If you get stuck on any problem in this part, you must go back and work the problems in Part I of this Supplement successfully before continuing.**

Note: Items 13-15 also occur in this module's *Model Exam*.

1. a. Define the momentum of a system of particles.

   b. Give the SI and English units of momentum.

   c. Express momentum units in terms of force and time units (SI and English). *Answer: 30*

2. a. Consider an isolated three-particle system. Use Newton's laws to show that the total momentum for this system does not change with time.

   b. Extend the argument of part (a) to an isolated N-particle system. *Answer: 31 Help: [S-1]*

3. The only external forces on a two-particle system act in the y-direction. Show that although this system is not isolated, the $\hat{x}$ and $\hat{z}$ components of the system momentum are constant in time. *Answer: 32*

4. Explain why a baseball player often moves his gloved hand in the direction of motion of the ball as he catches it. *Answer: 33*

5. A spinning rubber ball (weight = 0.64 lb) bounces from the floor as shown. The collision time is measured to be 0.010 s. Let $\hat{x}$ and $\hat{y}$ be horizontal and vertical unit vectors as shown.
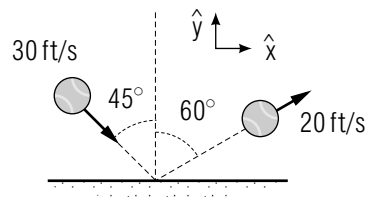
a. Calculate the ball's change in momentum during the collision, in terms of $\hat{x}$ and $\hat{y}$. *Help: [S-2]*

b. Determine the average resultant force on the ball during the collision. Express in terms of $\hat{x}$ and $\hat{y}$. *Help: [S-13]*

c. What objects exert force on the ball during the collision?

d. What is the average force on the floor during the collision? Express in terms of $\hat{x}$ and $\hat{y}$. *Help: [S-14]*

e. Is the floor frictionless? Explain. *Answer: 34*

6. Explain why a collision can be analyzed using momentum conservation even though the colliding objects do not constitute an isolated system. *Answer: 35*

7. Two masses which repel each other are released at $t = 0$ from rest on a horizontal frictionless surface. When the two are sufficiently separated, their interaction is negligible, and each is observed to move with a constant velocity. Let $\vec{w}$ be this constant velocity observed for m, and $\vec{V}$ the corresponding velocity for M.


a. What is the total momentum for this two mass system at $t = 0$?

b. Does this total momentum change for $t > 0$? Why?

c. Express the total momentum in terms of m, M, $\vec{w}$, and $\vec{V}$.

d. What can you deduce about the direction of $\vec{w}$ and $\vec{V}$? Explain.

e. Suppose the mass m is known and the velocities $\vec{w}$ and $\vec{V}$ are measured. Express M in terms of m, w, and V. *Answer: 36*

8. Showboat Steve is walking at 5.0 ft/s while Winsome Wylene's speed as she approaches him is 4.0 ft/s. Just before Showboat would have accelerated to pass in front of Winsome, they reach an icy (frictionless) patch on the walkway and suffer their inevitable collision. Holding desperately onto each other, they slide off together, hopefully to happier times.

If Showboat weighs 150 lb and Winsome 100.0 lb, determine their blissful velocity (magnitude and direction) after the collision. *Answer: 37 Help: [S-3]*

9. A child (mass = 40.0 kg) standing at rest on a smooth frozen lake throws a 0.60 kg ball toward a friend directly to the east. The initial speed of the ball is 10.0 m/s.

   a. Neglecting air resistance, what external forces act on the child-ball system during and after the throw?

   b. If the ball is moving horizontally just after release, determine the child's velocity (magnitude and direction) after release.

   c. If the ball's velocity immediately after release is 36.87° above horizontal, calculate the child's speed after release. *Answer: 38*

10. A 2.0 kg mass moving with a velocity of 15 m/s, east, collides with a 1.0 kg mass initially at rest. After the collision, the larger mass has a speed of 8.5 m/s and the smaller is observed to be moving with a speed of 17.0 m/s. Calculate the scattering angle for the larger mass; i.e., the angle between its final and initial velocities. *Answer: 39*
    Help: [S-4]

11. A mass M moving with a speed V in a particular direction collides with a larger object, of mass $3.0\,M$, that is initially at rest. After the collision the smaller mass is observed to be moving in a direction that makes an angle of 120° to its original direction, and it is now moving with a speed $V/(2.0)$. Determine the speed and direction of motion of the larger mass after the collision. Specify the direction of motion by giving the angle it makes to the pre-collision direction of motion of the smaller mass. *Answer: 40   Help: [S-5]*

12. A 5.0 kg mass moving with a velocity of $(3.2\hat{x} - 2.4\hat{y})$ m/s at $t = 0$ is acted upon by a force $\hat{x}F_x(t)+\hat{y}F_y(t)$ where $F_x(t)$ and $F_y(t)$ are shown in the graphs below. Determine the object's velocity at $t = 0.030$ s. *Answer: 41*

    Hint: Recall that $\Delta\vec{p} = \vec{F}_{R,av}\Delta t = \int_0^{\Delta t} \vec{F}_R(t)\,dt$.
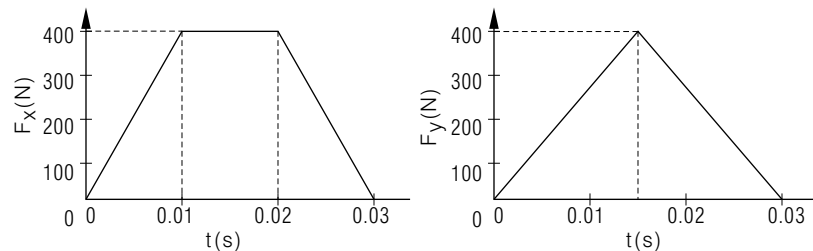
   Help: [S-7]

13. At an instant when a 0.20 kg particle has position, velocity, and acceleration given by $(2.1\hat{x} - 4.8\hat{y})$ m, $(1.2\hat{x} + 1.6\hat{z})$ m/s, and $(-2.0\hat{y} + 1.5\hat{z})$ m/s^2:

    a. Calculate its momentum.

    b. Calculate the rate at which its momentum is changing. *Answer: 42*

14. A diving competitor (weight = 160 lb) has a downward velocity of 11 ft/s just before hitting the board. When contact with the board ceases, 0.40 s later, the diver's velocity is 23 ft/s at an angle of 34° with the vertical. Calculate the magnitude of the average resultant force on the diver while in contact with the board. *Answer: 43   Help: [S-20]*

15. A mass M moving with a speed v collides with a mass $2M$ initially at rest. After the collision the two move as shown. Determine θ. *Answer: 44   Help: [S-10]*
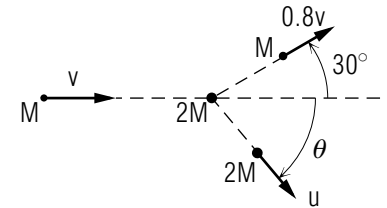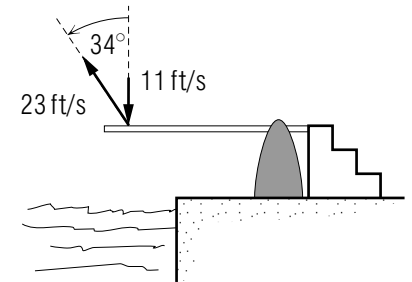
Answers.

1. The time elapsing between the instant of contact with the ground and the instant his downward motion stops.

2. $m(\vec{v}_{\text{final}} - \vec{v}_{\text{initial}}) = (79\,\text{kg}) \cdot (0\,\text{m/s} - 11\,\text{m/s, down}) = 869\,\text{kg m/s, up.}$

3. His average speed while stopping will be approximately one-half his initial speed; i.e., $v_{av} \simeq 5.5$ m/s. Hence his stopping time will be approximately $\Delta t \simeq x/v_{av}$, where x is the distance his "body" travels during the deceleration. In a full-knee bend, a body might fall approximately 0.8 m, so $\Delta t \simeq (0.8\,\text{m})/(5.5\,\text{m/s}) \simeq 0.1$ s.
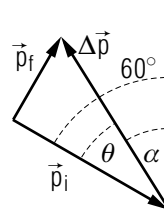
4. $\Delta\vec{p}/\Delta t \simeq (869\,\text{kg m/s, up})/(0.1\,\text{s}) = 9 \times 10^3\,\text{N}$, up.

5. $9000\,\text{N}/(79\,\text{kg} \times 9.8\,\text{m/s}^2) \simeq 10$.

6. $m = w/g = 100\,\text{slugs}$
   $p_i = mv_i = 6010\,\text{lb s}$
   $p_f = mv_f = 3230\,\text{lb s}$
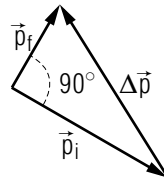
7. Since the angle between $\vec{p}_i$ and $\vec{p}_f$ is $90°$, the Pythagorean theorem gives
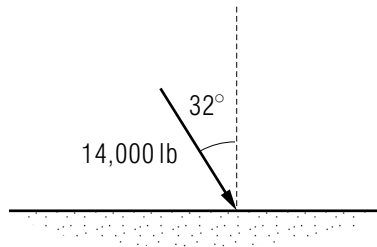
   $$|\Delta\vec{p}| = \sqrt{p_i^2 + p_f^2} = 6.8 \times 10^3\,\text{lb s}$$

8. $\alpha = 60° - \theta$
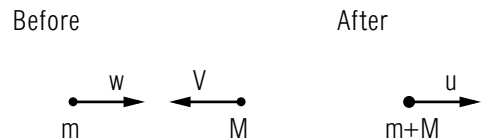 $\tan\theta = p_f/p_i = 0.537$
   $\theta = 28°$
   $\alpha = 32°$

9. $\dfrac{|\Delta\vec{p}|}{\Delta t} = 6823\,\text{lb s}/0.50\,\text{s} = 1.4 \times 10^4\,\text{lb}$

10.

11.

    Before   After

    Notice that in the sketch we have presumed the composite particle to

be moving to the right after the collision. Can you, in fact, determine the final direction of motion from what you are given? You will see shortly.

12. Since the stated problem makes no comment as to whether the two-particle system is isolated, you have no assurance that the system momentum is not changing. However, if we assume the time of the collision to be sufficiently short, then any momentum change by external forces will be negligible; and momentum conservation can then be used to relate momenta immediately before and after the collision. In any laboratory situation, the validity of such an assumption must always be considered carefully.

13. $mw\hat{x} - MV\hat{x} = (m + M)\vec{u}$.

    Did you miss the sign on the MV term?

    Don't forget it's moving in the negative $\hat{x}$ direction.

14. $(mw - MV/m + M)\,\hat{x}$.

    If $mw > MV$, $\vec{u}$ is in the $\hat{x}$ direction.

    If $mw = MV$, $\vec{u}$ is zero.

    If $mw < MV$, $\vec{u}$ is in the negative $\hat{x}$ direction.

15. greater than, equal in magnitude, less than.

16. If $V = 0$, $mw = (m+M)u$; or $m(w-u) = Mu$ and $M/m = (w-u)/u$.

17. $\dfrac{M}{m} = \dfrac{w-u}{u} \simeq \dfrac{w}{u} \gg 1$; or $M \gg m$?

18. If the two move off very slowly after the collision ($U \ll w$), then m must have collided with a much larger mass; e.g., a ping pong ball colliding with a locomotive at rest imparts verrrry little speed to the locomotive ... even on frictionless tracks! On the other hand, if an at-rest mass M were much smaller than an incoming mass m, the collision would have little effect on the speed of m. (Now you give the appropriate ping pong ball-locomotive analogy).

19. You cannot tell from the statement of the problem.

20. The system is isolated; i.e., the only force on m is that by M, and vice versa.

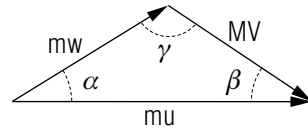21. If the collision time is sufficiently short, then external forces will not cause a measurable change in the system momentum during the collision.

22.
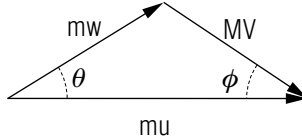
23. $\dfrac{A}{\sin\alpha} = \dfrac{B}{\sin\beta} = \dfrac{C}{\sin\gamma}$

24. $\dfrac{mw}{\sin\phi} = \dfrac{mu}{\sin\gamma}$

$w = \dfrac{\sin\phi}{\sin\gamma} u$

But $\gamma = \pi - (\theta + \phi)$

and $\sin(\pi - \theta - \phi) = \sin(\theta + \phi)$,

so $w = \dfrac{\sin\phi}{\sin(\theta + \phi)} u$.

25. $V = \dfrac{mu\sin\theta}{M\sin(\theta + \phi)}$

26. $mu = mw\cos\theta + MV\cos\phi$

$0 = mw\sin\theta - MV\sin\phi$

27. $MV = \dfrac{mw\sin\theta}{\sin\phi}$

$mu = mw\left[\cos\theta + \dfrac{\cos\phi\sin\theta}{\sin\phi}\right] = mw\left[\dfrac{\sin\phi\cos\theta + \cos\phi\sin\theta}{\sin\phi}\right]$

$mu = mw\dfrac{\sin(\phi + \theta)}{\sin\phi}$

$w = \dfrac{\sin\phi}{\sin(\phi + \theta)} u$

28. $MV = \dfrac{m\sin\theta}{\sin\phi}\dfrac{u\sin\phi}{\sin(\theta + \phi)} = \dfrac{mu\sin\theta}{\sin(\theta + \phi)}$

$V = \dfrac{mu\sin\theta}{M\sin(\theta + \phi)}$

29. $w = 21\,\text{m/s}$; $V = 1.0 \times 10^1\,\text{m/s}$.

30. a. $\vec{P} = \sum_k m_k\vec{v}_k$ where

$\vec{P}$ = momentum of system

m_k = mass of particle k

$\vec{v}_k$ = velocity of particle k

b. kg m/s; slug ft/s

c. N s ; lb s

31. a. $\vec{P} \equiv \vec{p}_1 + \vec{p}_2 + \vec{p}_3$

$d\vec{P}/dt = d\vec{p}_1/dt + d\vec{p}_2/dt + d\vec{p}_3/dt$

$= \vec{F}_{R1} + \vec{F}_{R2} + \vec{F}_R 3$

$= \vec{F}_{2,1} + \vec{F}_{3,1} + \vec{F}_{1,2} + \vec{F}_{3,2} + \vec{F}_{1,3} + \vec{F}_{2,3}$

$= (\vec{F}_{2,1} + \vec{F}_{1,2}) + (\vec{F}_{3,1} + \vec{F}_{1,3}) + (\vec{F}_{3,2} + \vec{F}_{2,3})$

$= 0.$

b. See this module's *Special Assistance Supplement*, [S-1].

32. $\vec{F} = d\vec{P}/dt$ may be written in scalar form as:

$F_x = dP_x/dt,\qquad F_y = dP_y/dt,\qquad F_z = dP_z/dr.$

Given $F_x = F_z = 0$, then $dP_x/dt = dP_z/dt = 0$ so P_x and P_z remain constant.

33. By letting the gloved hand move with the ball, he increases the "collision time" required to change the ball's momentum to zero, thereby reducing the average force on the ball, and hence (by Newton's third law) reducing the average force on his hand.

34. a. $(-0.078\hat{x} + 0.62\hat{y})\,\text{lb s}$.

b. $(-7.8\hat{x} + 62\hat{y})\,\text{lb}$.

c. floor, earth (weight).

d. $(7.8\hat{x} - 62\hat{y})\,\text{lb}$.

e. No. It exerts a horizontal force on the ball. A frictionless surface can exert normal forces only.

35. For most collisions the collision time is so small and consequently the collision forces so large, that the effect of external forces can be neglected.

36. a. Zero.

b. No, the resultant force on the system is zero.

c. $m\vec{w} + M\vec{V} = 0$.

d. In opposite directions since $\vec{V} = -(m/M)\vec{w}$.

e. $M = (w/V)\, m$.

37. $V = 3.4\,\text{ft/s}$.

$\theta = \tan^{-1}\left(\dfrac{1.6}{3.0}\right) = 28°$.
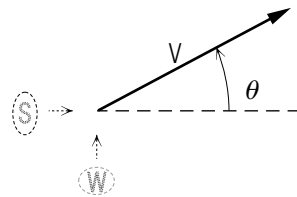
38. a. Weight (by earth) down on child and ball.

Upward force on child by ice.

b. $0.15\,\text{m/s}$, west.

c. $0.12\,\text{m/s}$, west.

39. 28°.

40. $\text{speed} = \dfrac{\sqrt{7.0}}{6.0}V = 0.44V$.

$\theta = \tan^{-1}\dfrac{\sqrt{3.0}}{5.0} = -19°$.

41. $(4.8\hat{x} - 1.2\hat{y})\,\text{m/s}$.

42. a. $(0.24\hat{x} + 0.32\hat{z})\,\text{kg m/s}$.

b. $(-0.40\hat{y} + 0.30\hat{z})\,\text{kg m/s}^2$.

43. $4.1 \times 10^2\,\text{lb}$.

44. $(5 \times 10^1)°$.

SPECIAL ASSISTANCE SUPPLEMENT

S-1   *(from PS-II-problem 2b)*

Note: answers are in [S-6].

1. If $\vec{p}_k$ is the momentum of particle k, then the total momentum $\vec{P}$ is given by: $\vec{P} = $ _____ .

2. Differentiating with respect to time gives:

$d\vec{P}/dt = \sum_{k=1}^{N}$ _____ .

3. According to Newton's second law, $d\vec{p}_k/dt = \vec{F}_{R,k}$ where $\vec{F}_{R,k}$ is _____ .

4. So $d\vec{P}/dt$ may be expressed in terms of the $\vec{F}_{R,k}$ as:

$d\vec{P}/dt = \vec{F}_{R,1} + $ _____ $= \sum_{k=1}^{N}$ _____

5. Since the system is isolated, each particle interacts with only the other $N - 1$ particles. Thus,

$\vec{F}_{R,1} = \vec{F}_{2\text{ on }1} + \vec{F}_{3\text{ on }1} + \ldots + \vec{F}_{N\text{ on }1}$

$\vec{F}_{R,2} = \vec{F}_{1\text{ on }2} + \vec{F}_{3\text{ on }2} + \ldots + \vec{F}_{N\text{ on }2}$

$\ldots$

$\vec{F}_{R,N} = \vec{F}_{1\text{ on }N} + \vec{F}_{3\text{ on }N} + \ldots + \vec{F}_{N-1\text{ on }N}$

where $\vec{F}_{j,k}$ is _____ .

6. $d\vec{P}/dt = \vec{F}_{2\text{ on }1} + \vec{F}_{3\text{ on }1} + \ldots + \vec{F}_{N\text{ on }1}$

$= +\vec{F}_{1\text{ on }2} + \vec{F}_{3\text{ on }2} + \ldots + \vec{F}_{N\text{ on }2} + \ldots$

$+ \vec{F}_{1\text{ on }N} + \vec{F}_{2\text{ on }N} + \ldots + \vec{F}_{N-1\text{ on }N}$ For each force $\vec{F}_{j\text{ on }k}$ in the sum above, there is a force $\vec{F}_{k\text{ on }j}$ which appears once and only once. If the sum on the right side of this equation is regrouped so that these paired forces are added (i.e., $\vec{F}_{j\text{ on }k} + \vec{F}_{k\text{ on }j}$), then each pair sums to _____ because of Newton's _____ law.

7. So $d\vec{P}/dt = $ _____ and $\vec{P}$ is _____ .

S-2   *(from PS-II-problem 5)*

You are probably subtracting the momentum vectors incorrectly. Here is a right way:

1. Calculate $\vec{p}_i$ and $\vec{p}_f$.

2. Write: $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{p}_f + (-\vec{p}_i)$.

3. Draw $\vec{p}_f$ and label it.

4. At the head of $\vec{p}_f$, start the tail of $(-\vec{p}_i)$.

5. Draw the rest of the vector $(-\vec{p}_i)$ (which is just $\vec{p}_i$ with the direction reversed) and label it.

6. Draw a vector connecting the tail of $\vec{p}_f$ to the head of $(-\vec{p}_i)$.

7. Label this new vector: $\Delta\vec{p} = \vec{p}_f + (-\vec{p}_i) = \vec{p}_f - \vec{p}_i$.

8. On the sketch, break $\Delta\vec{p}$ into its (approximate) x- and y-components. Note the sizes and signs of the two components.

Now calculate the x- and y-components of $\Delta\vec{p}$, noting that getting the contribution of $p_{i,y}$ involves taking the negative of a negative number.
 Help: [S-11]

S-3   *(from PS-II-problem 8)*

1. Did you properly convert each of the given weights to masses before putting them into the expression for momentum?

2. Did you calculate the total momentum, before the collision, using vector addition (getting its magnitude by taking the square root of the sum of the squares of the x- and y-components)?

3. Did you set the "after" momentum equal to the "before" momentum?

4. Did you set the "after" momentum equal to mv?

5. Did you remember that the total mass after the collision is just the sum of the masses of the two?

 Help: [S-8]

S-4   *(from PS-II-problem 10)*

Solving this problem symbolically involves a lot of algebra. We suggest solving this problem numerically. To do this, calculate the numerical magnitude of the *momentum* of each of the outgoing particles separately and draw the initial momentum and final momenta, roughly to scale, on a sketch of the scattering. Voila! For this special case the rest of the solution takes only a few lines.

S-5   *(from PS-II-problem 11)*

Apply conservation of momentum separately to the x- and y-components of momentum, leaving symbols for the (unknown) x- and y-components of the momentum of the larger mass after the collision. That gives you two equations. Solve for the two components, combine them to form a vector, and find the magnitude of the vector. Finally, divide that by the mass to get the speed. Get the angle from the ratio of the components.
 Help: [S-17]

S-6   *(from [S-1])*

Answers:

1. $\vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_N = \sum_{k=1}^{N} \vec{p}_k$

2. $d\vec{p}_k/dt$.

3. The resultant force on particle k.

4. $d\vec{P}/dt = \vec{F}_{R,1} + \vec{F}_{R,2} + \ldots + \vec{F}_{R,N} = \sum_{k=1}^{N} \vec{F}_{R,k}$.

5. The force particle j exerts on particle k.

6. Zero, third.

7. 0, constant (or conserved).

S-7   *(from PS-II-problem 12)*

Note that an integral between two points is the area under the curve between those same two points. *Help: [S-18]*

S-8 *(from [S-3])*

The x- and y-components of momentum are conserved separately.
Help: [S-15]

S-9 *(from PS-II-problem 14)*

The unit "lb" is a unit of weight, not mass.

S-10 *(from PS-II-problem 15)*

$Mv = M(0.8v)\cos 30° + 2M(u)\cos\theta$
$0 = M(0.8v)\sin 30° - 2M(u)\sin\theta$
Solve for θ. *Help: [S-12]*

S-11 *(from [S-2])*

$\vec{p}_i = \left(\dfrac{0.64\,\text{lb}}{32\,\text{ft/s}^2}\right)[(30\,\text{ft/s})\cos 45° \, \hat{x} - (30\,\text{ft/s})\sin 45° \, \hat{y}]$
That's enough help!

S-12 *(from [S-10])*

Divide all terms by (Mu), then replace (v/u) by the symbol a (an arbitrary symbol). That leaves two equations in two unknowns (a and θ). Solve for θ. To an unwarranted number of digits, $\theta = 52.5°$. *Further Help: [S-21]*

S-13 *(from PS-II-problem 5)*

If you claim "I don't know how to get the average force," then you did not pay attention to this module's text or even bother to skim over it to find the relevant material.

S-14 *(from PS-II-problem 5)*

Use Newton's third law.

S-15 *(from [S-8])*

$\vec{p}_s + \vec{p}_w = \left(\dfrac{150\,\text{lb}}{32\,\text{ft/s}^2}\right)(5.0\,\text{ft/s})\,\hat{x} + \left(\dfrac{100\,\text{lb}}{32\,\text{ft/s}^2}\right)(4.0\,\text{ft/s})\,\hat{y}$

$\vec{p}_{w+s} = \left(\dfrac{250\,\text{lb}}{32\,\text{ft/s}^2}\right)(v_x\,\hat{x} + v_y\,\hat{y})$

Now solve for v_x and v_y.

S-16 *(from AS-II-problem 9)*

Do problems 1-8 successfully before attempting this one.

S-17 *(from [S-5])*

$Mv = M(v/2)\cos 120° + 3.0MV\cos\theta$
$0 = M(v/2)\sin 120° + 3.0MV\sin\theta$
Solve for θ. *Help: [S-19]*

S-18 *(from [S-7])*

$\Delta v_x = \dfrac{1}{m}(400\,\text{N})(0.02\,\text{s}) = 1.6\,\text{N s}$

$\Delta v_y = \dfrac{1}{m}(400\,\text{N})(0.015\,\text{s}) = 1.2\,\text{N s}$

S-19 *(from [S-17])*

Divide all terms by (Mv), then replace (V/v) by the symbol a (an arbitrary symbol). That leaves two equations in two unknowns (a and θ). Solve for θ.
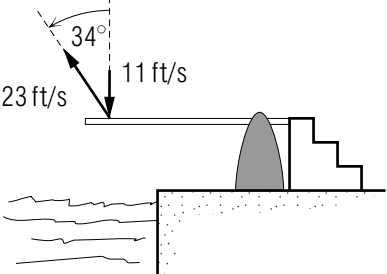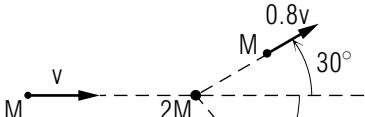

S-20 *(from PS-II-problem 14)*

$\vec{F}_{av} = \left(\dfrac{1}{.40\,\text{s}}\right)\left(\dfrac{160\,\text{lb}}{32\,\text{ft/s}^2}\right)[(23\,\text{ft/s})(\cos 34° \, \hat{y} + \sin 34° \, \hat{x}) - (-11\,\text{ft/s})\,\hat{y}]$
$= (376\,\hat{y} + 160\,\hat{x})\,\text{lb}$

S-21   *(from [S-12])*

Solving two equations that each contain the same two unknowns:

1. Solve one of the equations for either of the unknowns (pick one). Call it unknown #1.

2. Substitute that solution for unknown #1 into the other equation and solve for the other unknown. Call it unknown #2. Notice that the solution you just found for unknown #2 does not contain unknown #1.

3. Now substitute the solution for unknown #2 back into the solution for unknown #1. You now have solutions for both of the unknowns.

MODEL EXAM

1. See Output Skills K1-K2 in this module's *ID Sheet*. The actual exam may include one or both of these skills, or none.

2. At an instant when a 0.20 kg particle has position, velocity, and acceleration given by $(2.1\hat{x} - 4.8\hat{y})$ m, $(1.2\hat{x} + 1.6\hat{z})$ m/s, and $(-2.0\hat{y} + 1.5\hat{z})$ m/s^2:

   a. Calculate its momentum.

   b. Calculate the rate at which its momentum is changing.

3. A diving competitor (weight = 160 lb) has a downward velocity of 11 ft/s just before hitting the board. When contact with the board ceases, 0.40 s later, the diver's velocity is 23 ft/s at an angle of 34° with the vertical. Calculate the magnitude of the average resultant force on the diver while in contact with the board.

4. A mass M moving with a speed v collides with a mass $2M$ initially at rest. After the collision the two move as shown. Determine θ.

Brief Answers:

1. See this module's *text*.

2-4. See this module's *Problem Supplement*, problems 13-15.