MISN-0-13

LENGTH CONTRACTION AND TIME DILATION

$$x_{1R} - x_{2R} = k_{RL}\left[x_{1L} - x_{2L}\right]$$

$$\tau_L = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tau_R$$

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

LENGTH CONTRACTION AND TIME DILATION

by

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Title: **Length Contraction and Time Dilation**

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Input Skills:

1. Vocabulary: frame of reference (MISN-0-11), Lorentz transformation (MISN-0-12).
2. Transform single-event time and position measurements using the Lorentz transformation (MISN-0-12).

Output Skills (Knowledge):

K1. Vocabulary: laboratory frame, rest frame, length contraction, time dilation, twin paradox.

K2. Given the Lorentz transformation, derive the relativistic length contraction and time dilation factors.

K3. Show that the Lorentz transformation is independent of which frame of reference is considered to be moving and explain how this leads to the twin paradox.

Output Skills (Problem Solving):

S1. Given length and time intervals measured in one frame of reference, find the corresponding length and time intervals measured in a different frame of reference.

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1. Overview

The measured length of a moving object is always found to be shortened and the rate of a moving clock is found to be decreased. These effects are derived from the Lorentz transformation and are applied to physical systems. The symmetry of the Lorentz transformation is used to show the reasoning behind the simple form of the "twin paradox."

2. Length Contraction & Time Dilation

2a. The Length-Measurement Problem. The measurement of the length of an object which is traveling past you at high speed is not trivial.[1] You should put a stationary meter stick ahead of the object, parallel to its future path of travel. Then, when the object is alongside the meter stick, you must note the positions of the two ends simultaneously (just think of what would result otherwise). Subtracting the two end-position readings would give you the length (see Fig. 1).

2b. One Event in Two Frames. If an observer in one reference frame, A, and an observer in another reference frame, B, each measures the position and time of an event O, the position and time measurements

[1]For fifty years, physicists believed and taught that an object traveling past at high speed would appear physically distorted ("Lorentz contracted"). Then Terrell showed how it would really look. See "Appearances at Relativistic Speeds" (MISN-0-44).

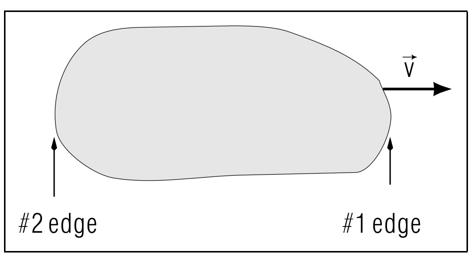

Figure 1. Defining an object's "length in the direction of travel."

in the two frames are related by the Lorentz transformation:

$$x_{OB}(t_{OB}) = k[x_{OA}(t_{OA}) - v_{BA}\, t_{OA}] \tag{1}$$

$$t_{OB} = k[t_{OA} - v_{BA}\, x_{OA}(t_{OA})/c^2] \tag{2}$$

where

$$k = [1 - v_{BA}^2/c^2]^{-1/2}\,.$$

In our current case, $x_{OA}(t_{OA})$ might refer to the location of one end of our[2] object as measured in the Lab frame at a particular Lab time. We say that the object is moving with respect to a Laboratory frame of reference L, and our task is to measure the moving object's length in its direction of motion, first in the Lab frame and then in a frame of reference which is moving along with the object. Since the object is at rest in the latter frame, that frame is called the object's Rest frame R.

2c. Derivation of the Length Contraction Factor. We label the leading edge of the object #1 and the trailing #2, as in the diagram. Then the Lab frame end-measurements are related to the Rest frame end-measurements by:

$$x_{1R}(t_{1R}) = k_{RL}[x_{1L}(t_{1L}) - v_{RL}\, t_{1L}]$$

$$x_{2R}(t_{2R}) = k_{RL}[x_{2L}(t_{2L}) - v_{RL}\, t_{2L}]\,.$$

As discussed in Sect. 2a, the Lab frame end-measurements must be simultaneous,

$$t_{1L} = t_{2L}\,,$$

because the object is moving. Subtracting the second equation from the first and using the simultaneity of measurement in the Lab frame,

$$x_{1R}(t_{1R}) - x_{2R}(t_{2R}) = k_{RL}[x_{1L}(t_L) - x_{2L}\,(t_L)]\,,$$

where t_L is the single time of measurement in the Lab.

Now in the Rest frame the object is not moving so measurements of the positions of the ends are independent of time, enabling us to write:

$$x_{1R} - x_{2R} = k_{RL}[x_{1L} - x_{2L}]\,.$$

Each of the above differences is the length as measured in the indicated frame and so we finally obtain:

$$\ell_{OR} = k_{RL}\ell_{OL}\,.$$

[2]The notation conventions are given in "Special Relativity: The Lorentz Transformation and the Velocity Addition Law" (MISN-0-12).

This is usually written the other way around:

$$\ell_{OL} = (1 - v_{RL}^2/c^2)^{1/2}\,\ell_{OR}\,. \tag{3}$$

Thus a moving object will always be measured to be shorter in the direction of travel than it will be when at rest and this length contraction is called the "Lorentz contraction."

2d. Derivation of the Time Dilation Factor. Now suppose we measure a time interval between two events. For concreteness we will follow a cosmic ray pion ("pī′-on"), denoted π, from its time of creation at time t_1 in the upper atmosphere to the end of its lifetime at time t_2 near the earth's surface. Our two frames of reference are the Rest frame of the π and the Lab frame which is stationary on the surface of the earth. We wish to compare the lifetimes of the π as observed from the two frames. From Eq. (2),

$$t_{1L} = k_{LR}[t_{1R} - v_{LR}\,x_{1R}(t_{1R})/c^2] \tag{4}$$

$$t_{2L} = k_{LR}[t_{2R} - v_{LR}\,x_{2R}(t_{2R})/c^2]\,. \tag{5}$$

To an observer in the Rest frame of the π, and π is not moving so its creation and annihilation are at the same space point:

$$x_{1R}(t_{1R}) = x_{2R}(t_{2R})\,. \tag{6}$$

Subtracting (4) from (5) and using (6) gives:

$$t_{2L} - t_{1L} = k_{LR}(t_{2R} - t_{1R})\,.$$

Since the time difference $t_2 - t_1$ is the lifetime τ of the pion, we finally get,

$$\tau_L = k\tau_R = (1 - v^2/c^2)^{-1/2}\,\tau_R\,.$$

If there is a frame of reference R in which two events occur at the same space point, that will be the frame in which the measured time interval between the events will be the smallest. Measured values by other observers will be larger by the factor k, and this is called the "time dilation" effect. Thus all slowly moving pions appear to have about the same lifetime while those moving at speeds near that of light appear to live longer, exactly in accordance with the factor k.

One can say that the fast moving pion appears to age more slowly!

3. The "Twin Paradox"

3a. The Paradox. The "twin paradox" refers to the differential aging of a set of twins, one of whom goes away on a space trip and eventually returns to earth. The twin who stayed at home is seen to be much older than the one who took the journey. You may or may not feel that this is truly a "paradox" but that is what it is called.

3b. *A*'s Space-time Point. The Lorentz transformation is independent of which frame is considered to be the moving one.

From twin A's point of view, twin B is moving with velocity v_{BA} so that:

$$x_{OB}(t_{OB}) = k[x_{OA}(t_{OA}) - v_{BA}t_{OA}] \tag{7}$$

$$t_{OB} = k[t_{OA} - v_{BA}x_{OA}(t_{OA})/c^2]\,. \tag{8}$$

However, from B's viewpoint, A is moving and has velocity $v_{AB} = -v_{BA}$. Hence we have:

$$x_{OA}(t_{OA}) = k[x_{OB}(t_{OB}) + v_{BA}\,t_{OB}] \tag{9}$$

$$t_{OA} = k[t_{OB} + v_{BA}\,x_{OB}(t_{OB})/c^2]\,. \tag{10}$$

Equations (9) and (10) are exactly equivalent to Eqs. (7) and (8). We show the equivalence by deriving Eqs. (7) and (8) from Eqs. (9) and (10).

3c. *B*'s Space Point. Solving Eq. (10) for t_{OB} gives,

$$t_{OB} = k^{-1}t_{OA} - v_{BA}x_{OB}(t_{OB})/c^2. \tag{11}$$

Substituting Eq. (11) into Eq. (9) gives,

$$\begin{aligned} x_{OA}(t_{OA}) &= k[x_{OB}(t_{OB}) + v_{BA}(k^{-1}\,t_{OA} - v_{BA}x_{OB}(t_{OB})/c^2] \\ &= k[k^{-1}v_{BA}t_{OA} + x_{OB}(t_{OB})(1 - v_{BA}^2/c^2)] \\ &= k[k^{-1}\,v_{BA}t_{OA} + k^{-2}x_{OB}(t_{OB})]\,. \end{aligned}$$

$$x_{OA}(t_{OA}) = v_{BA}t_{OA} + k^{-1}x_{OB}(t_{OB}) \tag{12}$$

Solving Eq. (12) for x_{OB} gives Eq. (7):

$$x_{OB}(t_{OB}) = k[x_{OA}(t_{OA}) - v_{BA}t_{OA}]\,.$$

3d. B's Event Time. We can similarly obtain Eq. (8) from Eqs. (9) and (10). We first solve Eq. (9) for $x_{OB}(t_{OB})$:

$$x_{OB}(t_{OB}) = k^{-1} x_{OA}(t_{OA}) - v_{BA} t_{OB}. \tag{13}$$

Then we use Eq. (13) to eliminate $x_{OB}(t_{OB})$ from Eq. (10), and obtain:

$$\begin{aligned} t_{OA} &= k\{t_{OB} + v_{BA}[k^{-1} x_{OA}(t_{OA}) - v_{BA} t_{OB}]/c^2\} \\ &= k[t_{OB}(1 - v_{BA}^2/c^2) + k^{-1} v_{BA} x_{OA}(t_{OA})/c^2] \\ &= k[k^{-2} t_{OB} + k^{-1} v_{BA} x_{OA}(t_{OA})/c^2] \end{aligned}$$

$$t_{OA} = k^{-1} t_{OB} + v_{BA} x_{OA}(t_{OA})/c^2 \tag{14}$$

Solving Eq. (14) for t_{OB} gives Eq. (8):

$$t_{OB} = k[t_{OA} - v_{BA} x_{OA}(t_{OA})/c^2].$$

3e. Time Interval Symmetry. Since the Lorentz transformation is symmetric for $A \leftrightarrow B$ (with $v_{BA} \leftrightarrow -v_{BA}$), all general results have that same symmetry. Thus if A sees B's clock running slower, B must see A's clock running slower. If A and B are otherwise the same ("twins") each sees the other age less rapidly. Thus twin A should see B as the younger and twin B should see A as the younger. Hence the paradox: how can each see the other as younger? Note, however, that in our derivation the twins could never come back together to compare their ages in the same frame of reference; we had no "turning around" or "stopping." The question of what would happen if one twin turned around, came back, and stopped by the other one, is not answered by Special Relativity: it has nothing to say about what happens during accelerations. For that one must go to the much more complex General theory of Relativity.[3]

Acknowledgments

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[3]See "The Equivalence Principle: An Introduction to Relativistic Gravitation" (MISN-0-110).

PROBLEM SUPPLEMENT

Problem 4 also occurs in this module's *Model Exam.*

1. A neutral pion, π^0, has a lifetime of 8.4×10^{-17} sec in its own rest frame before decaying into two photons. However in a laboratory experiment, where the pions are observed to be moving, their lifetime is measured to be 4.2×10^{-16} sec. Assuming this discrepancy to be due to time dilation, calculate the speed of the pions.

2. "There was a young fencer named Fisk, whose style was incredibly brisk. So fast was his action, that Lorentz space contraction foreshortened his foil to a disk." Assuming his 1.00 m long fencing foil only contracted to 10.0 cm (roughly the size of a good steak knife), how fast was Fisk lunging?

3. At the moment of the birth of a set of twins, one child is placed in a spaceship that rapidly accelerates to 0.866 c (0.866 times the speed of light), and travels to Alpha Centauri, 4.33 light years away (measured in the rest frame of the earth.)

   a. Calculate the age of each twin when the traveler reaches Alpha Centauri, as measured by the twin on earth.

   b. Calculate the distance from Alpha Centauri to earth, as measured by the traveling twin.

   c. How old is the traveling twin, as measured by himself, when he reaches Alpha Centauri?

   d. How old does the traveling twin perceive the earthbound twin to be?

4. Suppose you have a twin who is an astronaut. The twin travels at speed 0.9999 c to the vicinity of a star which is 60 light years away from earth (one light year is the distance light travels in one year).

   a. Find your age and your twin's age at the time you observe that your twin reaches the star.

   b. Show that the length, time and velocity observed by your twin check.

Brief Answers:

1. $v = 0.98\,\mathrm{c}$
2. $v = 0.995\,\mathrm{c}$
3. a. Age of stationary twin: 5 yrs.; Age of traveling twin: 2.5 yrs.
   b. 2.165 light years
   c. 2.5 yrs.
   d. 1.25 yrs.
4. a. Present age +60.006 yr., present age +0.85 yr.
   Twin observes distance of 0.848507 light years, time of 0.84859 yr., velocity of −0.9999 c.
   b. Then (0.848507/0.84859) c = 0.9999 c: check.

MODEL EXAM

$$x_{\mathrm{OB}}(t_{\mathrm{B}}) = k[x_{\mathrm{OA}}(t_{\mathrm{A}}) - v_{\mathrm{BA}}\, t_{\mathrm{A}}]$$

$$t_{\mathrm{B}} = k[t_{\mathrm{A}} - v_{\mathrm{BA}}\, x_{\mathrm{OA}}(t_{\mathrm{A}})/c^2]$$

$$k = [1 - v_{\mathrm{BA}}^2/c^2]^{-1/2}$$

1. See Output Skills K1-K3 in this module's *ID Sheet*.
2. Suppose you have a twin who is an astronaut. The twin travels at speed 0.9999 c to the vicinity of a star which is 60 light years away from earth (one light year is the distance light travels in one year).
   a. Find your age and your twin's age at the time you observe that your twin reaches the star.
   b. Show that the length, time and velocity observed by your twin check.

Brief Answers:

1. See text of Module.
2. See this module's *Problem Supplement*, problem 4.